ISPT notes

Martin Lüscher December 2014

1. Introduction

These notes contain some technical material related to the construction of trivializing stochastic fields described in ref. [1]. Equations in that paper are referred to by equation numbers in square brackets.

2. Statistical weight of Feynman diagrams

Let $(\mathcal{V},\mathcal{L})$ be a pair of vertex and line lists describing a Feynman diagram that contributes to the correlation function [3.6]. In this section, the statistical weight of the diagram is derived from the functional integral using Wick's theorem.

2.1 Preliminaries

Without loss, the vertex list $\mathcal{V}$ may be assumed to be such that the boson vertices are listed before the fermion-boson vertices. Moreover, the vertices in each of these subsets may be unambiguously ordered according to their order in the coupling and the number of boson legs.

When the functional integral is expanded in powers of the coupling, the diagram described by the lists $(\mathcal{V},\mathcal{L})$ is obtained by contracting the fields in a certain product of fields in particular ways. The product includes the fields

$$\varphi_{a_1}\ldots\varphi_{a_n}\psi_{\alpha_1}\ldots\psi_{\alpha_m}\overline{\psi}_{\beta_m}\ldots\overline{\psi}_{\beta_1} \tag{2.1}$$

in the correlation function [3.6] and a factor

$$\frac{1}{p!}\left\{-\frac{1}{j!}V^{(k,j)}_{b_1\ldots b_j}\varphi_{b_1}\cdots\varphi_{b_j}\right\}^p \tag{2.2}$$

for each type of boson vertex in the list $\mathcal{V}$ (and similarly for each type of fermion-boson vertex), where p is the multiplicity of the vertex in the list.

2.2 Relating Wick contractions to diagram data

The vertex factors in the product of fields may be labeled in the order of the vertices in the list $\mathcal{V}$. Each contraction of a pair of fields then determines a line item (b, i_1, i_2) or (f, i_1, i_2), where i_1, i_2 are the indices of the vertex factors containing the fields or of the fields themselves if they are contained in the product (2.1). A contraction C of all fields thus yields a list $\mathcal{L}_C$ of line items, which is unique up to T_2 transformations. Together with the vertex list $\mathcal{V}$, $\mathcal{L}_C$ describes the Feynman diagram representing the contraction.

In the following, attention will be restricted to contractions C, where the diagram is the one described $(\mathcal{V}, \mathcal{L})$, i.e. where $(\mathcal{V}, \mathcal{L}_C)$ is related to $(\mathcal{V}, \mathcal{L})$ by a transformation T_1 followed by a transformation T_2. Since both diagram data have the same vertex list, the transformation T_1 can only permute the indices of the vertices of equal type. Some of the permutations moreover change the line list in a way that can be compensated by a transformation T_2. The number of these special transformations is equal to the factor F_1 of the symmetry factor of the diagram.

2.3 Counting contractions

It follows from these remarks that the contractions C divide into disjoint classes labeled by the transformations T_1 modulo T_1 symmetry transformations. The number of classes,

$$n_{\mathrm{cl}} = \frac{p_1!p_2!\ldots p_s!}{F_1}, \tag{2.3}$$

is determined by the symmetry factor F_1 and the multiplicities $p_1, \ldots, p_s$ of the different vertex symbols in the vertex list. Moreover, the number n_{c} of contractions in each class is the same for all classes, since the contractions in any two classes are related by a relabeling of vertex factors of the same type in the product of fields. The total number of contractions is therefore equal to $n_{\mathrm{cl}}n_{\mathrm{c}}$.

In order to determine n_{c}, one may consider the class of contractions C where $\mathcal{L}_C$ coincides with $\mathcal{L}$ modulo transformations T_2. The pairing of fields in the product of fields is then determined by the line list $\mathcal{L}$ up to permutations of the boson fields in the vertex factors. The number of these permutations is $j_1! \dots j_{n_v}!$, where j_i denotes the number of boson fields contained in the i'th vertex factor. Not all permutations correspond to different pairings of fields. In particular, if l fields in a vertex factor are contracted with l fields in another vertex factor, the number of distinct pairings is only $l!$ rather than $(l!)^2$. There are also only $(2l-1)!!$ instead of $(2l)!$ pairings if $2l$ fields in a vertex factor are contracted with themselves. With respect to the number of field permutations, the number

$$n_\mathrm{c} = \frac{j_1! \dots j_{n_v}!}{F_2 F_3} \tag{2.4}$$

of contractions is therefore reduced by the product of the symmetry factors F_2 and F_3 of the diagram.

2.4 Synthesis

The statistical factor of the diagram is the product of the number of contractions represented by the diagram times the product of the factors

$$\frac{1}{p!(j!)^p} \tag{2.5}$$

that come with the vertex factors (2.2). Recalling eqs. (2.3) and (2.4), the statistical factor is thus found to coincide with the inverse of the symmetry factor $F_1 F_2 F_3$ of the diagram.

3. Efficient contraction of products of diagrams

The number of contractions of the random fields attached to the vertices of the tree diagrams $\mathcal{R}_1, \dots, \mathcal{R}_n$ in eq. [7.3] grows rapidly with the order of the diagrams. In practice, it is therefore advisable to collect contractions that obviously yield the same ordinary diagram.

3.1 Problem description

Let $\mathcal{V}$ be an ordered list of n_v vertices with $j_1, \ldots, j_{n_v}$ attached random boson fields $\theta_{k,a}$ with fixed index k. There are thus

$$N = \sum_{i=1}^{n_v} j_i \tag{3.1}$$

such fields, which may be labeled, in arbitrary order, by an index h running from 1 to N. Other random fields may be attached to the vertices as well, but can be ignored in the following since they only contract with their own kind.

A Wick contraction C of $2l$ of these random boson fields is fully specified by a list of pairs

$$[(h_1, h_2), (h_3, h_4), \ldots, (h_{2l-1}, h_{2l})] \tag{3.2}$$

of the indices that label the fields. Up to reorderings of the pairs, or of the indices within the pairs, there are

$$\binom{N}{2l}(2l-1)!! \tag{3.3}$$

different contractions ($2l \leq N$ is assumed without further notice).

The Wick contraction (3.2) corresponds to a line list

$$\mathcal{L}_C = [(\mathrm{b}, i_1, i_2), (\mathrm{b}, i_3, i_4), \ldots, (\mathrm{b}, i_{2l-1}, i_{2l})] \tag{3.4}$$

where i_k is the index of the vertex to which the field number h_k is attached. In general, this is a many-to-one mapping, i.e. there may be several Wick contractions that yield the same line list modulo transformations T_2. The goal in this appendix is to enumerate the different line lists and to count the Wick contractions corresponding to each of them.

3.2 Number of contractions

Let C be a contraction and $\mathcal{L}_C$ the associated line list as described in the previous subsection. The number l_i of lines attached to the vertex number i is in the range $0 \leq l_i \leq j_i$ and may be less than j_i if $2l < N$.

Since the random fields are indistinguishable, the contraction effectively selects l_i fields out of j_i fields at the vertex number i. In total there are

$$\binom{j_1}{l_1}\ldots\binom{j_{n_v}}{l_{n_v}} \tag{3.5}$$

different selections one can make and the number of contractions with line list $\mathcal{L}_C$ is therefore proportional to this factor.

The other factors can be determined following the lines of the second paragraph in subsect. 2.3. As a result one obtains

$$\binom{j_1}{l_1}\ldots\binom{j_{n_v}}{l_{n_v}}\frac{l_1!\ldots l_{n_v}!}{F_2F_3} \tag{3.6}$$

for the number of contractions, where F_2 and F_3 are the symmetry factors of the diagram defined by the vertices and the line list $\mathcal{L}_C$ (cf. subsection 4.3 of ref. [1]).

3.3 Enumerating line lists

Without loss, the vertex indices may be assumed to run from 1 to n_v. Moreover, the indices of the endpoints in a line (b, k, l) may be required to be ordered so that $k \leq l$. A total ordering of the lines is then defined by

$$(k_1, l_1) \leq (k_2, l_2) \quad \Leftrightarrow \quad k_1 < k_2 \ \text{ or } \ (k_1 = k_2 \ \text{ and } \ l_1 \leq l_2). \tag{3.7}$$

The ordering ambiguity in the line lists (3.4) may then be eliminated by requiring the lines to be in ascending order.

Ordered line lists with l lines can be generated straightforwardly through l nested loops over the set of all possible lines, where the loop variable in the outermost loop is the first line, the one in the next-to-outermost loop the second line, and so on. The i'th loop must start at the current value of the $(i-1)$'th loop variable in order to ensure that properly ordered line lists are obtained. Moreover, the line lists where too many lines end at any vertex must be dropped. The algorithm must thus keep track of the numbers of occupied external lines and avoid cycling through large sets of line lists that are eventually dropped.

3.4 Synthesis

The random fields at the leaves of the tree diagrams $\mathcal{R}_1, \ldots, \mathcal{R}_n$ may thus be contracted by first making an inventory of the random fields, where fields of the same type are collected in groups together with information of where they are attached.

A generator of ordered line lists is then set up for each group of fields and these are run in a nested loop so that all possible combinations of line lists are obtained.

A given combination of ordered line lists represents a set of contractions of the random fields. Each contraction is contained in one and only one set. The number of contractions in a given set is equal to the product of the factors (3.6) (one factor per type of random field). In general, these factors are not small and cycling through ordered line lists can thus be much more efficient than generating all contractions.

References

[1] M. Lüscher, *Instantaneous stochastic perturbation theory*, CERN-PH-TH-2014-239