Lattice QCD vertices in ISPT

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1. Introduction

In instantaneous stochastic perturbation theory (ISPT), the representative stochastic fields are given by a sum of tree diagrams with random fields attached to their leaves [1]. The numerical evaluation of these diagrams requires the vertices of the theory to be programmed up to the desired order in the coupling.

In this note, the computation of the vertices is discussed in the case of Wilson's formulation of lattice QCD. Both SF [2] and open-SF [3] boundary conditions are considered.

2. Action

The lattice theory is set up as in ref. [3]. Further details, particularly on the gauge-fixing [4], are given in the notes [5]. Only the Wilson gauge action is considered and all $\mathrm{O}(a)$-improvement boundary counterterms are set to their tree-level values (thus $c_1 = 0$ and $c_\mathrm{G} = c'_\mathrm{G} = 1$).

2.1 Gauge action

Let $\mathcal{S}_0$ be the set of *oriented* plaquette loops on the lattice. The Wilson gauge action is then given by

$$S_\mathrm{G} = \frac{1}{g_0^2} \sum_{\mathcal{C} \in \mathcal{S}_0} w_0(\mathcal{C}) \,\mathrm{tr}\{1 - U(\mathcal{C})\}, \tag{2.1}$$

where g_0 denotes the bare coupling and $U(\mathcal{C})$ the ordered product of the link variables around the plaquette loop $\mathcal{C}$. The weight factor $w_0(\mathcal{C})$ is equal to $\frac{1}{2}$ for the space-like loops $\mathcal{C}$ at time 0 and equal to 1 in all other cases.

In perturbation theory, the link variables

$$U(x,\mu) = \exp\{g_0 A_\mu(x)\}, \qquad A_\mu(x) = A_\mu^a(x) T^a, \tag{2.2}$$

are represented through a gauge potential $A_\mu^a(x)$. The change of integration variables in the functional integral from the link variables to the gauge potential is associated with a Jacobian that amounts to adding the term

$$S_\mathrm{m} = -\sum_{x,\mu} \mathrm{tr}\{\ln[J(g_0 A_\mu(x))]\} \tag{2.3}$$

to the total action. In this expression, $J(X)$ denotes the linear operator acting on the Lie algebra $\mathfrak{su}(N)$ of the gauge group, which represents the differential of the exponential map (see appendix A).

2.2 Gauge-fixing and ghost action

As discussed in the notes [5], a natural choice of the gauge-fixing action in the case of open-SF boundary conditions is

$$S_\mathrm{gf} = \tfrac{1}{2}\lambda_0 \sum_{x_0=0}^{T-1}{}' \sum_{\vec{x}} \partial_\mu^* A_\mu^a(x) \partial_\nu^* A_\nu^a(x) \tag{2.4}$$

together with the prescription

$$A_0(x-\hat{0}) = -A_0(x) \quad \text{at} \quad x_0 = 0. \tag{2.5}$$

The primed summation symbol in eq. (2.4) implies that the terms at $x_0 = 0$ are counted with weight $\frac{1}{2}$ and all other terms with weight 1.

When SF boundary conditions are chosen, the gauge-fixing action

$$S_\mathrm{gf} = \tfrac{1}{2}\lambda_0 \left\{ \sum_{x_0=1}^{T-1} \sum_{\vec{x}} \partial_\mu^* A_\mu^a(x) \partial_\nu^* A_\nu^a(x) + \frac{1}{T^3 L^3} \sum_{y,z} A_0^a(y) A_0^a(z) \right\} \tag{2.6}$$

includes two terms, where the second fixes the residual (constant) gauge transformations at time $x_0 = 0$ (in ref. [2] the gauge was fixed in a slightly different way).

In all cases, the associated Faddeev-Popov ghost action assumes the form

$$S_{\mathrm{FP}} = \sum_{x_0=0}^{T-1} \sum_{\vec{x}} \partial_0 \bar{c}^a(x) \big\{ J(g_0 A_0(x))^{-1} \partial_0 + g_0 \mathrm{Ad}\, A_0(x) \big\} c^a(x)$$

$$+ \sum_{x_0=0}^{T-1}{}' \sum_{\vec{x}} \sum_{k=1}^{3} \partial_k \bar{c}^a(x) \big\{ J(g_0 A_k(x))^{-1} \partial_k + g_0 \mathrm{Ad}\, A_k(x) \big\} c^a(x) \tag{2.7}$$

with the boundary conditions

$$c(x) = \bar{c}(x) = 0 \quad \text{at} \quad x_0 = T \tag{2.8}$$

and additionally

$$\partial_k c(x) = \partial_k \bar{c}(x) = 0 \quad \text{at} \quad x_0 = 0 \tag{2.9}$$

if SF boundary conditions are chosen [5].

3. Vertex functions

3.1 Definition

The expansions of the various actions in powers of the coupling are of the general form

$$S_{\mathrm{G}} = \sum_{n=2}^{\infty} \frac{g_0^{n-2}}{n!} \sum_{x_1,\ldots,x_n} V_{\mathrm{G}}^{(n)}(x_1,\ldots,x_n)^{a_1\ldots a_n}_{\mu_1\ldots\mu_n} A^{a_1}_{\mu_1}(x_1)\ldots A^{a_n}_{\mu_n}(x_n), \tag{3.1}$$

$$S_{\mathrm{m}} = \sum_{n=2}^{\infty} \frac{g_0^{n}}{n!} \sum_{x_1,\ldots,x_n} V_{\mathrm{m}}^{(n)}(x_1,\ldots,x_n)^{a_1\ldots a_n}_{\mu_1\ldots\mu_n} A^{a_1}_{\mu_1}(x_1)\ldots A^{a_n}_{\mu_n}(x_n), \tag{3.2}$$

$$S_{\mathrm{FP}} = \sum_{n=0}^{\infty} \frac{g_0^{n}}{n!} \sum_{x_1,\ldots,x_n,y_1,y_2} V_{\mathrm{FP}}^{(n)}(x_1,\ldots,x_n,y_1,y_2)^{a_1\ldots a_n,b_1,b_2}_{\mu_1\ldots\mu_n}$$

$$\times A^{a_1}_{\mu_1}(x_1)\ldots A^{a_n}_{\mu_n}(x_n) \bar{c}^{b_1}(y_1) c^{b_2}(y_2), \tag{3.3}$$

where repeated Lorentz and Lie algebra indices are summed over. The time coordinates are summed from 0 to $T-1$ without any weight factors. All vertex functions $V_{\rm G}^{(n)}$, $V_{\rm m}^{(n)}$ and $V_{\rm FP}^{(n)}$ are required to be symmetric under permutations of the arguments $(x_1,\mu_1,a_1),\ldots,(x_n,\mu_n,a_n)$. In the case of open-SF boundary conditions, the vertex functions are then uniquely determined by the expansions (3.1)–(3.3).

These vertex functions also satisfy eqs. (3.1)–(3.3) if SF boundary conditions are chosen. The actions $S_{\rm G}$, $S_{\rm m}$ and $S_{\rm FP}$ actually do not explicitly refer to the boundary conditions and only the boundary values of the fields at time $x_0=0$ depend on them. In the present context, where the vertex functions are ultimately contracted with fields having the correct boundary values, it is therefore consistent to use the same expressions for the vertex functions for both open-SF and SF boundary conditions.

3.2 Programs for the vertices

For the evaluation of the tree diagrams contributing to the trivializing stochastic field, the vertex functions are required in the form

$$
\begin{aligned}
(\mathcal{A}_{\rm G}^{(n)})_\mu^a(x) = -\sum_{x_1,\ldots,x_{n-1}} & V_{\rm G}^{(n)}(x,x_1,\ldots,x_{n-1})_{\mu,\mu_1\ldots\mu_{n-1}}^{a,a_1\ldots a_{n-1}} \\
& \times(\mathcal{A}_1)_{\mu_1}^{a_1}(x_1)\ldots(\mathcal{A}_{n-1})_{\mu_{n-1}}^{a_{n-1}}(x_{n-1}),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
(\mathcal{A}_{\rm m}^{(n)})_\mu^a(x) = -\sum_{x_1,\ldots,x_{n-1}} & V_{\rm m}^{(n)}(x,x_1,\ldots,x_{n-1})_{\mu,\mu_1\ldots\mu_{n-1}}^{a,a_1\ldots a_{n-1}} \\
& \times(\mathcal{A}_1)_{\mu_1}^{a_1}(x_1)\ldots(\mathcal{A}_{n-1})_{\mu_{n-1}}^{a_{n-1}}(x_{n-1}),
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
(\mathcal{A}_{\rm FP}^{(n)})_\mu^a(x) = -\sum_{x_1,\ldots,x_{n-1},y_1,y_2} & V_{\rm FP}^{(n)}(x,x_1,\ldots,x_{n-1},y_1,y_2)_{\mu,\mu_1\ldots\mu_{n-1}}^{a,a_1\ldots a_{n-1},b_1,b_2} \\
& \times(\mathcal{A}_1)_{\mu_1}^{a_1}(x_1)\ldots(\mathcal{A}_{n-1})_{\mu_{n-1}}^{a_{n-1}}(x_{n-1})\overline{\mathcal{C}}^{b_1}(y_1)\mathcal{C}^{b_2}(y_2),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
(\mathcal{C}_{\rm FP}^{(n)})^a(x) = -\sum_{x_1,\ldots,x_n,y} & V_{\rm FP}^{(n)}(x_1,\ldots,x_n,x,y)_{\mu_1\ldots\mu_n}^{a_1\ldots a_n,a,b} \\
& \times(\mathcal{A}_1)_{\mu_1}^{a_1}(x_1)\ldots(\mathcal{A}_n)_{\mu_n}^{a_n}(x_n)\mathcal{C}^b(y),
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
(\overline{\mathcal{C}}_{\rm FP}^{(n)})^a(x) = -\sum_{x_1,\ldots,x_n,y} & V_{\rm FP}^{(n)}(x_1,\ldots,x_n,y,x)_{\mu_1\ldots\mu_n}^{a_1\ldots a_n,b,a} \\
& \times(\mathcal{A}_1)_{\mu_1}^{a_1}(x_1)\ldots(\mathcal{A}_n)_{\mu_n}^{a_n}(x_n)\overline{\mathcal{C}}^b(y),
\end{aligned}
\tag{3.8}
$$

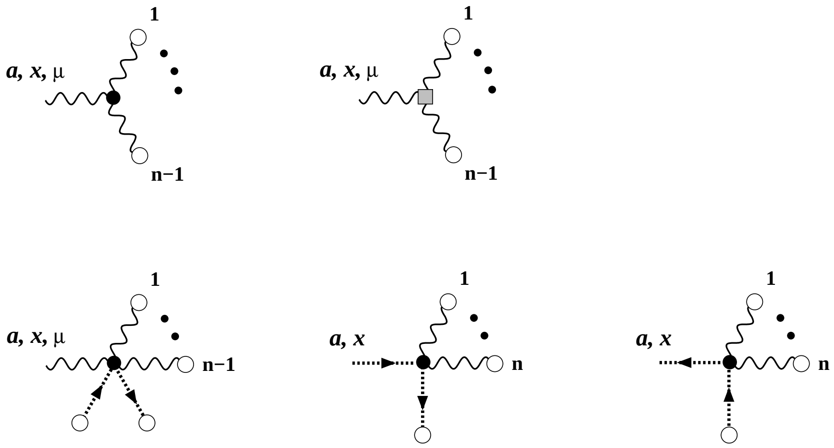

Fig. 1. Graphical representation of the fields (3.4)–(3.8). The open circles stand for the source fields $\mathcal{A}_1,\ldots,\mathcal{A}_n$, $\mathcal{C}$ and $\overline{\mathcal{C}}$ (cf. ref. [1]; the powers of g_0 multiplying the vertices are omitted here).

where the fields on the right of these equations are *complex* source fields satisfying the chosen boundary conditions (see fig. 1). When a vertex is to be evaluated, the source fields are already known. The programs that compute the fields (3.4)–(3.8) thus take as input the order n of the vertex and the addresses of the source fields.

4. Computation of the functions $\mathcal{A}_{\mathrm{G}}^{(n)}$

In order to simplify the notation, the coupling g_0 is set to unity in the following. For the reasons explained at the end of subsect. 3.1, only the case of open-SF boundary conditions is considered from now on. The gauge potential may take values in the complexified Lie algebra $\mathfrak{sl}(N,\mathbb{C})$ of the gauge group [1].

4.1 Alternative expression for $\mathcal{A}_{\mathrm{G}}^{(n)}$

The definition of the vertex functions $V_{\mathrm{G}}^{(n)}$ and the field $\mathcal{A}_{\mathrm{G}}^{(n)}$ imply that

$$(\mathcal{A}_{\mathrm{G}}^{(n)})_\mu^a(x) = -\left\{\frac{\partial^{n-1}}{\partial t_1\ldots\partial t_{n-1}}\left(\frac{\partial S_{\mathrm{G}}}{\partial A_\mu^a(x)}\right)\left(\textstyle\sum_{k=1}^{n-1} t_k\mathcal{A}_k\right)\right\}_{t_1=\ldots=t_{n-1}=0}. \tag{4.1}$$

The expression on the right of this equation may be evaluated by running through all unoriented plaquettes on the lattice and by collecting their contributions to the field

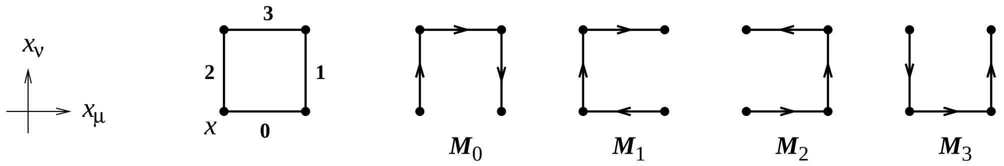

Fig. 2. Labeling of the edges of the plaquettes in the (μ, ν)-plane. The orientation of the associated staple matrices M_l is chosen so as to ensure that eq. (4.2) holds for all $l = 0, \ldots, 3$.

$\mathcal{A}_{\rm G}^{(n)}$. Let $\mathcal{C}$ be the plaquette loop in the (μ, ν)-plane with lower-left corner x. The field variables and associated staple matrices on the plaquette are labeled as shown in fig. 2. If A_l denotes the gauge potential on the link number l, the contribution of the plaquette to the gauge action may be written in a form

$$-w_0(\mathcal{C})\mathrm{tr}\left\{\exp\{A_l\}M_l + \exp\{-A_l\}M_l^{-1}\right\} + \text{constant}, \tag{4.2}$$

where the dependence of the contribution on A_l is made explicit.

The derivative with respect to the gauge potential may now be worked out using the identities

$$\frac{\partial \exp\{\pm A_\mu(x)\}}{\partial A_\mu^a(x)} = \pm \exp\{\pm A_\mu(x)\} J\big(\pm A_\mu(x)\big) T^a \tag{4.3}$$

and

$$\mathrm{e}^X J(X) Y = \sum_{i,j=0}^{\infty} \frac{1}{(i+j+1)!} X^i Y X^j. \tag{4.4}$$

For the contribution of the plaquette to the field $\mathcal{A}_{\rm G}^{(n)}$ on the link number l, the expression

$$w_0(\mathcal{C}) \sum_{i,j=0}^{\infty} \frac{1}{(i+j+1)!}$$
$$\times \frac{\partial^{n-1}}{\partial t_1 \ldots \partial t_{n-1}} \mathrm{tr}\left\{T^a (A_l)^i \left(M_l - (-1)^{i+j} M_l^{-1}\right)(A_l)^j\right\}\bigg|_{t_1 = \ldots = t_{n-1} = 0} \tag{4.5}$$

is thus obtained. In this and the following equations, it is understood that the gauge potential in the trace is replaced by the sum $\sum_{k=1}^{n-1} t_k \mathcal{A}_k$ of the source fields. The sum over i, j is effectively finite, since the summand vanishes when $i + j \geq n$.

4.2 Multiple differentiation algebra

The differential operators in eq. (4.5) potentially generate many terms. To be able to organize the computation efficiently, it is worth going through some further notation and basic algebraic facts.

Let P be a subset of the index set $P_0 = \{1, 2, \ldots, n-1\}$. For any differentiable (possibly matrix-valued) function $f(t)$ of the parameters $t_1, \ldots, t_{n-1}$ the operator

$$\partial_P f = \left\{ \left[\textstyle\prod_{k \in P} \partial_{t_k} \right] f(t) \right\}_{t_1 = \ldots = t_{n-1} = 0} \tag{4.6}$$

may be defined. In particular,

$$\partial_P f = f(0) \;\text{ if }\; P = \emptyset. \tag{4.7}$$

The subsets P of P_0 may be labeled by an index ranging from 0 to $2^{n-1} - 1$ exactly like the symmetric matrix products $\{\ldots\}_{\mathrm{s}}$ introduced in appendix B. The differentiation of the link variables,

$$\partial_P \exp\{\pm A_\rho(z)\} =$$

$$\begin{cases} (\pm 1)^p \left\{ (\mathcal{A}_{i_1})_\rho(z) \ldots (\mathcal{A}_{i_p})_\rho(z) \right\}_{\mathrm{s}} & \text{if } P = \{i_1, \ldots, i_p\} \neq \emptyset, \\ 1 & \text{if } P = \emptyset, \end{cases} \tag{4.8}$$

actually establishes a one-to-one correspondence between the differential operators and the matrix products of the source fields on the link considered.

Another important formula is the Leibniz rule

$$\partial_P (fg) = \sum_{Q \subset P} \partial_Q f \, \partial_{P \setminus Q} g \tag{4.9}$$

for the differentiation of products of functions (and matrices). The number of subsets Q of P is equal to $2^{|P|}$, where $|P|$ denotes the number of elements of P. There are thus that many terms in the sum (4.9) and

$$\sum_{p=0}^{n-1} \binom{n-1}{p} 2^p = 3^{n-1} \tag{4.10}$$

products $\partial_Q f \, \partial_{P \setminus Q} g$ to be computed if all derivatives $\partial_P (fg)$ are to be calculated.

4.3 Differentiation with respect to $t_1, .., t_{n-1}$

Using the Leibniz rule, eq. (4.5) may be rewritten in the form

$$w_0(\mathcal{C}) \sum_{Q_1,Q_2\subset P_0,Q_1\cap Q_2=\emptyset} \frac{1}{(n-|Q_2|)!}$$

$$\times\mathrm{tr}\left\{T^a \partial_{Q_1} V_l\, \partial_{Q_2}\bigl(M_l + (-1)^{n-|Q_2|} M_l^{-1}\bigr)\, \partial_{P_0\backslash(Q_1\cup Q_2)} V_l\right\}, \tag{4.11}$$

$$V_l = \sum_{k=0}^{n-1} (A_l)^k. \tag{4.12}$$

The sums over i and j in eq. (4.5) are implicitly included in the two factors V_l, but since

$$\partial_Q V_l = \partial_Q (A_l)^{|Q|} \tag{4.13}$$

each of these sums actually reduces to a single term when the differential operators are applied. For the same reason, $i + j$ could be replaced by $n - 1 - |Q_2|$.

The expression (4.11) may be evaluated by rewriting it in the form

$$w_0(\mathcal{C})\mathrm{tr}\left\{T^a \sum_{P\subset P_0}\left[\sum_{Q\subset P} \frac{1}{(n-|Q|)!} \partial_{P\backslash Q} V_l\, \partial_Q\bigl(M_l + (-1)^{n-|Q|} M_l^{-1}\bigr)\right]\partial_{P_0\backslash P} V_l\right\} \tag{4.14}$$

and by computing the sums, first the one over Q for all $P \subset P_0$ and then the sum over P. Before these sums can be performed, the matrices $\partial_Q\bigl(M_l + (-1)^{n-|Q|} M_l^{-1}\bigr)$ must be calculated for all $Q \subset P_0$ following the lines of appendix C.

Per plaquette, and not counting multiplications by matrices proportional to the unit matrix, the calculation requires

$$16\left(3^{n-1} - 2^n + 1\right) + 4\,(n + 1)\left(2^{n-2} - 1\right) \tag{4.15}$$

$N \times N$ matrix multiplications to be performed. For $n = 4$, 6 and 8, for example, (4.15) evaluates to 252, 3300 and 33180. These figures are quite large, but in the case of interest, $N = 3$, current processor cores can do some 5×10^7 matrix multiplications per second. Assuming a local lattice of size 16×8^3, the time needed for all matrix multiplications is then 0.25, 3.2 and 33 seconds, respectively.

4.4 3-point vertex

There are two source fields in this case and many terms in the sum (4.11) cancel. In particular, all terms where $|Q_2| < 2$ vanish. One is then left with the term

$$w_0(\mathcal{C})\,\mathrm{tr}\left\{T^a \partial_{P_0}\big(M_l - M_l^{-1}\big)\right\}. \tag{4.16}$$

Moreover, non-zero terms are obtained only when the derivatives with respect to t_1 and t_2 act on different link variables in the staple matrices.

For all $k > l$ in $\{0, 1, 2, 3\}$ let

$$X_{kl} = [(\mathcal{A}_1)_k\,, (\mathcal{A}_2)_l] + [(\mathcal{A}_2)_k\,, (\mathcal{A}_1)_l]\,. \tag{4.17}$$

In terms of these commutators

$$\partial_{P_0}\big(M_0 - M_0^{-1}\big) = X_{32} + X_{21} + X_{31}, \tag{4.18}$$

$$\partial_{P_0}\big(M_1 - M_1^{-1}\big) = X_{32} - X_{20} - X_{30}, \tag{4.19}$$

$$\partial_{P_0}\big(M_2 - M_2^{-1}\big) = X_{10} - X_{30} - X_{31}, \tag{4.20}$$

$$\partial_{P_0}\big(M_3 - M_3^{-1}\big) = X_{10} + X_{20} + X_{21}. \tag{4.21}$$

Per plaquette the calculation thus requires 24 matrix multiplication and 26 matrix additions.

5. Computation of the functions $\mathcal{A}_{\mathrm{m}}^{(n)}$

5.1 Expansion of the action S_{m}

For any matrix $X \in \mathfrak{sl}(N, \mathbb{C})$, the identities

$$\mathrm{tr}\left\{\ln J(X)\right\} = \sum_{n=1}^{\infty} \frac{B_n}{n n!} \mathrm{tr}\left\{(\mathrm{Ad}\,X)^n\right\}, \tag{5.1}$$

$$\mathrm{tr}\left\{(\mathrm{Ad}\,X)^n\right\} = -\delta_{n0} + \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathrm{tr}\left\{X^k\right\} \mathrm{tr}\left\{X^{n-k}\right\} \tag{5.2}$$

hold, where $B_0, B_1, \ldots$ are the Bernoulli numbers. Since $\mathrm{Ad}\, X$ is an antisymmetric matrix, the odd-power terms in the series (5.1) vanish.

The action (2.3) is therefore given by

$$S_{\mathrm{m}} = -\sum_{x,\mu}\sum_{n=2,4,\ldots}\sum_{k=0}^{n}(-1)^k \frac{B_n}{nk!(n-k)!} \mathrm{tr}\left\{(A_\mu(x))^k\right\} \mathrm{tr}\left\{(A_\mu(x))^{n-k}\right\} \tag{5.3}$$

and the functions $\mathcal{A}_{\mathrm{m}}^{(n)}$ vanish if n is odd (as in section 4, the coupling g_0 is set to unity). In the rest of this section, n is assumed to be even and positive.

5.2 Differentiation of the action

As in the case of the gauge vertices, the function $\mathcal{A}_{\mathrm{m}}^{(n)}$ may be obtained through

$$(\mathcal{A}_{\mathrm{m}}^{(n)})_\mu^a(x) = -\left\{\frac{\partial^{n-1}}{\partial t_1 \ldots \partial t_{n-1}}\left(\frac{\partial S_{\mathrm{m}}}{\partial A_\mu^a(x)}\right)\left(\textstyle\sum_{k=1}^{n-1} t_k \mathcal{A}_k\right)\right\}_{t_1=\ldots=t_{n-1}=0}. \tag{5.4}$$

The program that computes the function for given source fields visits the links (x,μ) one by one and sets

$$(\mathcal{A}_{\mathrm{m}}^{(n)})_\mu^a(x) = \sum_{k=2}^{n}(-1)^k \frac{2B_n}{n(k-1)!(n-k)!}$$
$$\times\, \partial_{P_0}\left[\mathrm{tr}\left\{T^a (A_\mu(x))^{k-1}\right\} \mathrm{tr}\left\{(A_\mu(x))^{n-k}\right\}\right]. \tag{5.5}$$

This leads to

$$(\mathcal{A}_{\mathrm{m}}^{(n)})_\mu^a(x) = \frac{2B_n}{n}\sum_{P\subset P_0} \mathrm{tr}\left\{T^a \partial_P \exp\{A_\mu(x)\}\right\} \mathrm{tr}\left\{\partial_{P_0\setminus P} \exp\{-A_\mu(x)\}\right\} \tag{5.6}$$

when the Leibniz rule is applied.

5.3 2-point vertex

In this case, there is a single non-zero term on the right of eq. (5.6) and the result

$$(\mathcal{A}_{\mathrm{m}}^{(2)})_\mu(x) = -\frac{N}{12}(\mathcal{A}_1)_\mu(x) \tag{5.7}$$

is quickly obtained.

6. Computation of the functions $\mathcal{A}_{\rm FP}^{(n)}$

6.1 Expansion of the ghost action

The ghost action (2.7) may be written in the form of a sum

$$S_{\rm FP} = -2\sum_{x,\mu} w_\mu(x) {\rm tr}\left\{\partial_\mu \bar{c}(x)\left[J(A_\mu(x))^{-1}\partial_\mu c(x) + {\rm Ad}\, A_\mu(x) c(x)\right]\right\} \tag{6.1}$$

over the lattice links (x,μ), where

$$w_\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x_0 = 0 \text{ and } \mu > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{6.2}$$

Recalling eq. (A.13) and using the identity

$$({\rm Ad}\, X)^n Y = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} X^k Y X^{n-k}, \tag{6.3}$$

the action is readily expanded in powers of the gauge potential. The function $\mathcal{A}_{\rm FP}^{(n)}$ is then obtained as usual by differentiation with respect to the gauge potential and by applying the operator ∂_{P_0}.

6.2 3-point vertex

The function

$$(\mathcal{A}_{\rm FP}^{(1)})_\mu(x) = w_\mu(x)\left[\partial_\mu \overline{\mathcal{C}}(x), \tfrac{1}{2}\partial_\mu \mathcal{C}(x) + \mathcal{C}(x)\right] \tag{6.4}$$

is the only one to which the last term in eq. (6.1) contributes. Use has here been made of the fact that the ghost source fields are complex fields rather than fields with values in a Grassmann algebra.

6.3 Higher-order vertices

Since $B_{2m+1} = 0$ for $m \geq 1$, all higher-order vertices with an odd number of gluon legs vanish. In the rest of this section, n is therefore assumed to be even and positive.

Combining eqs. (6.1),(6.3) and (A.13), the expression

$$(\mathcal{A}_{\rm FP}^{(n)})_\mu^a(x) = 2w_\mu(x) B_n \sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!} \sum_{j=0}^{k-1} \partial_{P_0} {\rm tr}\big\{T^a A_\mu(x)^{k-j-1}$$

$$\times\big[\partial_\mu\mathcal{C}(x)A_\mu(x)^{n-k}\partial_\mu\overline{\mathcal{C}}(x)A_\mu(x)^j + \partial_\mu\overline{\mathcal{C}}(x)A_\mu(x)^{n-k}\partial_\mu\mathcal{C}(x)A_\mu(x)^j\big]\big\} \quad (6.5)$$

is obtained. Setting

$$V_\mu(x) = \sum_{i=0}^{n-1}(A_\mu(x))^i, \quad (6.6)$$

the differentiation in eq. (6.5) can be carried out in two steps. First

$$\partial_P Z_\mu(x) = \sum_{Q\subset P}\frac{(-1)^{|Q|}}{|Q|!(n-|Q|)!}\big[\partial_\mu\mathcal{C}(x)\partial_Q V_\mu(x)\partial_\mu\overline{\mathcal{C}}(x)\partial_{P\setminus Q}V_\mu(x)$$
$$+\,\partial_\mu\overline{\mathcal{C}}(x)\partial_Q V_\mu(x)\partial_\mu\mathcal{C}(x)\partial_{P\setminus Q}V_\mu(x)\big] \quad (6.7)$$

is computed for all $P \subset P_0$ and then the field

$$(\mathcal{A}_{\mathrm{FP}}^{(n)})_\mu^a(x) = w_\mu(x)2B_n\sum_{P\subset P_0}\mathrm{tr}\left\{T^a\partial_{P_0\setminus P}V_\mu(x)\partial_P Z_\mu(x)\right\}. \quad (6.8)$$

If well organized, the computation requires

$$2\cdot 3^{n-1} + (n+5)(2^{n-2}-1) + 3 \quad (6.9)$$

matrix multiplications to be done per link.

7. Computation of the functions $\mathcal{C}_{\mathrm{FP}}^{(n)}$ and $\overline{\mathcal{C}}_{\mathrm{FP}}^{(n)}$

7.1 3-point vertices

These vertices are given by

$$\mathcal{C}_{\mathrm{FP}}^{(1)}(x) = \sum_{\mu=0}^{3} w_\mu(x)\partial_\mu^*\left\{\left[(\mathcal{A}_1)_\mu(x), \tfrac{1}{2}\partial_\mu\mathcal{C}(x)+\mathcal{C}(x)\right]\right\}, \quad (7.1)$$

$$\overline{\mathcal{C}}_{\mathrm{FP}}^{(1)}(x) = \sum_{\mu=0}^{3} w_\mu(x)\left(\left[(\mathcal{A}_1)_\mu(x), \partial_\mu\overline{\mathcal{C}}(x)\right] - \tfrac{1}{2}\partial_\mu^*\left\{\left[(\mathcal{A}_1)_\mu(x), \partial_\mu\overline{\mathcal{C}}(x)\right]\right\}\right), \quad (7.2)$$

where the expressions in the curly brackets are set to zero at $x_0 = -1$.

7.2 Higher-order vertices

In the following, n is again assumed to be even and positive. There are n source fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in this case, and in the expressions below the gauge potential stands for the linear combination $\sum_{k=1}^{n} t_k \mathcal{A}_k$. The index set P_0 accordingly gets replaced by $P_1 = \{1, 2, \ldots, n\}$.

With these notational conventions, the fields to be computed are

$$\mathcal{C}_{\mathrm{FP}}^{(n)}(x) = \sum_{\mu=0}^{3} w_\mu(x) \partial_\mu^* \mathcal{V}_\mu(x), \qquad \mathcal{V}_\mu(x) = \partial_{P_1} J(A_\mu(x))^{-1} \partial_\mu \mathcal{C}(x), \tag{7.3}$$

$$\overline{\mathcal{C}}_{\mathrm{FP}}^{(n)}(x) = \sum_{\mu=0}^{3} w_\mu(x) \partial_\mu^* \overline{\mathcal{V}}_\mu(x), \qquad \overline{\mathcal{V}}_\mu(x) = \partial_{P_1} J(A_\mu(x))^{-1} \partial_\mu \overline{\mathcal{C}}(x), \tag{7.4}$$

where, by definition,

$$\partial_0^* \mathcal{V}_0(x)\big|_{x_0=0} = \mathcal{V}_0(x), \qquad \partial_0^* \overline{\mathcal{V}}_0(x)\big|_{x_0=0} = \overline{\mathcal{V}}_0(x). \tag{7.5}$$

Recalling eqs. (A.13) and (6.3), the expressions

$$\mathcal{V}_\mu(x) = B_n \sum_{P \subset P_1} \partial_P \exp\{A_\mu(x)\} \partial_\mu \mathcal{C}(x) \partial_{P_1 \setminus P} \exp\{-A_\mu(x)\}, \tag{7.6}$$

$$\overline{\mathcal{V}}_\mu(x) = B_n \sum_{P \subset P_1} \partial_P \exp\{A_\mu(x)\} \partial_\mu \overline{\mathcal{C}}(x) \partial_{P_1 \setminus P} \exp\{-A_\mu(x)\}, \tag{7.7}$$

are then obtained.

8. Gauge-fixing vertex

The gauge-fixing vertex must be included in the Feynman rules if one is interested in the renormalized perturbation expansion of gauge-variant correlation functions. It can also serve as probe for the gauge-invariance of the results obtained for gauge-invariant observables.

Unlike the other vertex functions, the gauge-fixing vertex function explicitly depends on the boundary conditions. First a field $F(x)$ with values in $\mathfrak{sl}(3,\mathbb{C})$ needs to be computed. For $0 \leq x_0 < T$ and open-SF boundary conditions, the field is given by

$$F(x) = \partial_\mu^* A_\mu(x), \tag{8.1}$$

where the rule (2.5) is to be applied at $x_0 = 0$. In the case of SF boundary conditions,

$$F(x)|_{x_0=0} = \frac{1}{T^2 L^3} \sum_y A_0(y), \tag{8.2}$$

$$F(x)|_{x_0>0} = (1 - x_0/T)F(0) + \partial_\mu^* A_\mu(x). \tag{8.3}$$

For both boundary conditions, it is convenient to set $F(x)|_{x_0=T} = 0$. The function is thus contained in the space of infinitesimal gauge transformations and actually coincides with the gauge-fixing function $(\eth A)(x)$ introduced in [5].

Once $F(x)$ is computed, the gauge-fixing vertex function is obtained through

$$(\mathcal{A}_{\rm gf})_\mu^a(x) = w_\mu(x)\partial_\mu F^a(x), \tag{8.4}$$

the weight factor $w_\mu(x)$ being given by eq. (6.2) as before.

Appendix A

A.1 Lie algebra

The Lie algebra $\mathfrak{su}(N)$ of $\mathrm{SU}(N)$ may be identified with the space of complex $N \times N$ matrices X satisfying

$$X^\dagger = -X, \qquad \mathrm{tr}\{X\} = 0. \tag{A.1}$$

With respect to an orthonormal basis

$$T^a \in \mathfrak{su}(N), \quad a = 1, \ldots, N^2 - 1,$$

$$\mathrm{tr}\{T^a T^b\} = -\tfrac{1}{2}\delta^{ab}, \tag{A.2}$$

of such matrices, the general element X of the Lie algebra is given by $X^a T^a$ with real coefficients X^a. The structure constants f^{abc}, defined through

$$[T^a, T^b] = f^{abc} T^c, \tag{A.3}$$

are real too and totally anti-symmetric under permutations of the indices.

A.2 Adjoint representation

The representation space of the adjoint representation of $\mathfrak{su}(N)$ is the Lie algebra itself, i.e. the elements X of $\mathfrak{su}(N)$ are represented by linear transformations

$$\mathrm{Ad}\, X : \mathfrak{su}(N) \mapsto \mathfrak{su}(N). \tag{A.4}$$

Explicitly, $\mathrm{Ad}\, X$ is defined through

$$\mathrm{Ad}\, X \cdot Y = [X, Y] \quad \text{for all} \quad Y \in \mathfrak{su}(N). \tag{A.5}$$

With respect to a basis T^a the associated matrix $(\mathrm{Ad}\, X)^{ab}$ representing the transformation is given by

$$\mathrm{Ad}\, X \cdot T^b = T^a (\mathrm{Ad}\, X)^{ab}, \tag{A.6}$$

which is equivalent to

$$(\mathrm{Ad}\, X)^{ab} = -f^{abc} X^c, \qquad (\mathrm{Ad}\, X \cdot Y)^a = f^{abc} X^b Y^c, \tag{A.7}$$

in terms of the structure constants. In particular, $(\mathrm{Ad}\, X)^{ab}$ is an anti-symmetric matrix.

A.3 Differential of the exponential mapping

Let X be an element of $\mathfrak{su}(N)$. A linear mapping $J(X) : \mathfrak{su}(N) \mapsto \mathfrak{su}(N)$ may then be defined through

$$J(X) \cdot Y = \mathrm{e}^{-X} \left. \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{X+tY} \right|_{t=0} \qquad \text{for all} \quad Y \in \mathfrak{su}(N). \tag{A.8}$$

$J(X)$ is referred to as the differential of the exponential mapping. It is possible to show that

$$J(X) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} (\mathrm{Ad}\, X)^k, \tag{A.9}$$

which may symbolically be written as

$$J(X) = \frac{1 - \mathrm{e}^{-\mathrm{Ad}\,X}}{\mathrm{Ad}\,X}. \tag{A.10}$$

The associated matrix $J(X)^{ab}$, representing the transformation through

$$J(X) \cdot T^b = T^a J(X)^{ab}, \tag{A.11}$$

is real and satisfies

$$J(X)^T = J(-X), \qquad J(X)^{-1} - J(-X)^{-1} = \mathrm{Ad}\,X. \tag{A.12}$$

In particular, the expansion

$$J(X)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{k!} (\mathrm{Ad}\,X)^k, \tag{A.13}$$

$$B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_3 = 0,\ \ldots \qquad \text{(the Bernoulli numbers)}, \tag{A.14}$$

has no odd terms beyond the first order term.

Appendix B

Let $X_1, \ldots, X_m$ be complex square matrices and $i_1, \ldots, i_p$ a sequence of p pairwise different integers in the range $1, \ldots, m$. Define the symmetric product

$$\left\{X_{i_1} \ldots X_{i_p}\right\}_{\mathrm{s}} = \frac{1}{p!} \sum_{\sigma \in S_p} X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(p)}}, \tag{B.1}$$

where S_p denotes the symmetric group of p elements. There are

$$\sum_{p=1}^{m} \binom{m}{p} = 2^m - 1 \tag{B.2}$$

symmetric products of this kind.

The products can be computed recursively through

$$\left\{X_{i_1}\dots X_{i_p}\right\}_{\rm s} = \frac{1}{p}\sum_{k=1}^{p} X_{i_k}\left\{X_{i_1}\dots X_{i_{k-1}}X_{i_{k+1}}\dots X_{i_p}\right\}_{\rm s}. \tag{B.3}$$

For the computation of all symmetric products, the recursion requires

$$\sum_{p=2}^{m} p\binom{m}{p} = m\left(2^{m-1}-1\right) \tag{B.4}$$

matrix multiplications to be performed.

Symmetric matrix products may be labeled by an integer i in the range $[1, 2^m - 1]$. If

$$i = b_m b_{m-1}\dots b_1 \tag{B.5}$$

is the binary representation of i through the bits $b_1,\dots,b_m$, the factors X_k included in the associated symmetric product (B.1) are the ones where $b_k = 1$. The number of the factors is thus equal to the bit count of the index i.

Appendix C

The matrices

$$\partial_Q\bigl(M_l + (-1)^{n-|Q|}M_l^{-1}\bigr) \tag{C.1}$$

can be computed in two steps. First the matrices

$$\partial_P Y_{kl} = (-1)^{|P|}\sum_{R\subset P}\partial_R \exp\{A_k\}\,\partial_{P\setminus R}\exp\{A_l\} \tag{C.2}$$

are computed for $(k,l)\in\{(0,1),(1,0),(2,3),(3,2)\}$. In the second step, the products

$$\partial_Q\bigl(M_0 + (-1)^{n-|Q|}M_0^{-1}\bigr) =$$

$$\sum_{P\subset Q}\left\{\partial_{Q\setminus P}\exp\{A_1\}\,\partial_P Y_{32} + (-1)^n\partial_P Y_{23}\,\partial_{Q\setminus P}\exp\{A_1\}\right\}, \tag{C.3}$$

$$\partial_Q\bigl(M_1+(-1)^{n-|Q|}M_1^{-1}\bigr)=$$

$$\sum_{P\subset Q}\bigl\{\partial_P Y_{32}\,\partial_{Q\setminus P}\exp\{A_0\}\,+(-1)^n\partial_{Q\setminus P}\exp\{A_0\}\,\partial_P Y_{23}\bigr\}\,, \tag{C.4}$$

$$\partial_Q\bigl(M_2+(-1)^{n-|Q|}M_2^{-1}\bigr)=$$

$$\sum_{P\subset Q}\bigl\{\partial_{Q\setminus P}\exp\{A_3\}\,\partial_P Y_{10}+(-1)^n\partial_P Y_{01}\,\partial_{Q\setminus P}\exp\{A_3\}\bigr\}\,, \tag{C.5}$$

$$\partial_Q\bigl(M_3+(-1)^{n-|Q|}M_3^{-1}\bigr)=$$

$$\sum_{P\subset Q}\bigl\{\partial_P Y_{10}\,\partial_{Q\setminus P}\exp\{A_2\}+(-1)^n\partial_{Q\setminus P}\exp\{A_2\}\,\partial_P Y_{01}\bigr\}\,, \tag{C.6}$$

are evaluated.

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