Numerical stochastic perturbation theory and the Langevin equation

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December 2015; revised January 2016

1 Introduction

In these notes we describe an implementation of numerical stochastic perturbation theory for the pure SU(N) Yang-Mills theory based on the Langevin equation [1,2,3,4,5]. In particular, some specific integrator and the necessary gauge damping are discussed in detail. The presentation closely follows the one of ref. [6], to which we refer to, as well as to ref. [7], for the definition of the lattice set-up and unexplained notation.

2 Langevin equation

2.1 Case without gauge damping

2.1.1 Definition

The pure SU(N) Yang-Mills theory on the lattice is defined in terms of the link variables $U(x,\mu) \in \mathrm{SU}(N)$. The Langevin equation,

$$\partial_t U_t(x,\mu) = [-w_{x,\mu}(\partial^a_{x,\mu} S_G)(U_t)T^a + \eta_t(x,\mu)]U_t(x,\mu), \tag{2.1}$$

$$\langle \eta^a_t(x,\mu)\rangle = 0, \qquad \langle \eta^a_t(x,\mu)\eta^b_s(y,\nu)\rangle = 2w_{x,\mu}\delta^{ab}\delta_{\mu\nu}\delta(t-s)\delta_{xy}, \tag{2.2}$$

then determines the trajectory in field space, $U_t(x,\mu)$, along the stochastic time t, given some initial value for the gauge field at a given time (say $t=0$). In these equations, $\eta_t(x,\mu) = \eta^a_t(x,\mu)T^a$ is a Gaussian random noise field, while $\partial^a_{x,\mu}S_G(U_t)$ denotes the derivative of the gauge action S_G with respect to the link variable $U_t(x,\mu)$ in the direction of the SU(N) generator T^a. Note in particular that a weight factor $w_{x,\mu}$ is included. The latter depends on the chosen boundary conditions. Specifically, for SF boundary conditions this reads,

$$w_{x,\mu} = \begin{cases} 0, & \text{if } x_0 = 0 \text{ and } \mu > 0, \\ 1, & \text{otherwise,} \end{cases} \tag{2.3}$$

while for open-SF boundary conditions,

$$w_{x,\mu} = \begin{cases} 2, & \text{if } x_0 = 0 \text{ and } \mu > 0, \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

This weight factor ensures that the Langevin equation is diagonal in momentum space at leading order in the coupling [6] (see also Appendix A). As we shall see shortly, however, it does not modify the equilibrium distribution of the stochastic process under consideration, but rather influences the approach to it. Before presenting this result, we also note that in order to obtain a consistent perturbative expansion of the Langevin equation, one needs to rescale the stochastic time as $t \to tg_0^2$ in (2.1) and (2.2). As a result one can consider instead of (2.1) the equation,

$$\partial_t U_t(x,\mu) = [-g_0^2 w_{x,\mu}(\partial^a_{x,\mu} S_G)(U_t)T^a + g_0 \eta_t(x,\mu)]U_t(x,\mu), \tag{2.5}$$

while leaving (2.2) unchanged.

2.1.2 Equilibrium distribution

Following the steps of [8,9], it is easy to derive the Fokker-Planck (FP) equation associated to the Langevin equation (2.5). The result is given by,

$$\partial_t P_t(U) = \sum_{x,\mu,a} g_0^2 w_{x,\mu}\, \partial^a_{x,\mu}\big[(\partial^a_{x,\mu} S_G)(U) + \partial^a_{x,\mu}\big] P_t(U), \tag{2.6}$$

where $P_t(U)$ is the probability distribution of the gauge field U at time t. In particular, the equilibrium distribution, $P(U)$, corresponds to a fixed point of the FP equation and satisfies,

$$\sum_{x,\mu,a} g_0^2 w_{x,\mu}\, \partial^a_{x,\mu}\big[(\partial^a_{x,\mu} S_G)(U) + \partial^a_{x,\mu}\big] P(U) = 0. \tag{2.7}$$

Clearly, a solution to this equation is,

$$P(U) \propto e^{-S_G(U)}. \tag{2.8}$$

As anticipated, the boundary weight factor does not influence the equilibrium distribution. (The same is trivially true of course for the rescaling of the stochastic time.) This freedom in defining the Langevin equation is in fact a well known feature of the dynamics (see e.g. [10]).

2.2 Case with gauge damping

2.2.1 Definition

In a numerical implementation of stochastic perturbation theory, the introduction of a gauge damping is necessary in order to stabilize the simulations. This can be obtained by considering a time dependent gauge transformation of the fields,

$$U_t(x,\mu) \to \Lambda_t(x) U_t(x,\mu) \Lambda_t(x+\hat{\mu})^{-1}, \qquad \eta_t(x,\mu) \to \Lambda_t(x)\eta_t(x,\mu)\Lambda_t(x)^{-1}. \tag{2.9}$$

Note that in order to be consistent with the boundary conditions the gauge transformation needs to satisfy the conditions,

$$\Lambda_t(x)|_{x_0=T} = 1, \qquad \Lambda_t(x+\hat{k}L) = \Lambda_t(x), \tag{2.10}$$

in the case of open-SF boundary conditions, and additionally,

$$\partial_k \Lambda_t(x)|_{x_0=0} = 0, \quad k = 1, 2, 3, \tag{2.11}$$

if SF boundary conditions are chosen. By convention we also set

$$\Lambda_t(x)|_{t=0} = 1. \tag{2.12}$$

Taking into account the Langevin equation (2.5), it is then easy to show that the gauge transformed fields satisfy the modified equation,

$$\partial_t U_t(x,\mu) = [-g_0^2 w_{x,\mu}(\partial^a_{x,\mu} S_G)(U_t)T^a - g_0 \nabla_\mu \omega_t(x) + g_0 \eta_t(x,\mu)]U_t(x,\mu), \tag{2.13}$$

where,

$$\omega_t(x) = \frac{1}{g_0}\partial_t \Lambda_t(x)\Lambda_t(x)^{-1} \in \mathfrak{su}(N), \tag{2.14}$$

and where we introduced the gauge covariant derivative,

$$\nabla_\mu \omega_t(x) = U_t(x,\mu)\omega_t(x+\hat{\mu})U_t(x,\mu)^{-1} - \omega_t(x). \tag{2.15}$$

The defining equations for the noise (2.2), instead, are unchanged.

Given the modified Langevin equation a damping of the longitudinal modes of the gauge field can be obtained by choosing [6] (see also Appendix A),

$$\omega_t(x) = -\lambda_0 \sum_{\mu=0}^{3} \partial^*_\mu C_t(x,\mu), \tag{2.16}$$

and

$$\omega_t(x) = -\lambda_0 \Big\{ (1 - x_0/T)\frac{1}{T^2 L^3}\sum_y C_t(y,0) + \sum_{\mu=0}^{3} \partial^*_\mu C_t(x,\mu) \Big\}, \tag{2.17}$$

for open-SF and SF boundary conditions, respectively. In these equations,

$$C_t(x,\mu) = \frac{1}{2g_0}\Big\{ U_t(x,\mu) - U_t(x,\mu)^{-1} - \frac{1}{N}\mathrm{tr}\big[U_t(x,\mu) - U_t(x,\mu)^{-1}\big] \Big\}, \tag{2.18}$$

takes values in $\mathfrak{su}(N)$ and,

$$\partial^*_0 C_t(x,0)|_{x_0=0} = \begin{cases} 2C_t(x,0), & \text{for open-SF bc,} \\ 0, & \text{for SF bc,} \end{cases} \tag{2.19}$$

by convention.

2.2.2 Equilibrium distribution

The FP equation associated to the modified Langevin equation (2.13) reads (cf. eq. (2.6)),

$$\partial_t P_t(U) = \sum_{x,\mu,a} \partial^a_{x,\mu} \big[g_0^2 w_{x,\mu} (\partial^a_{x,\mu} S_G)(U) + g_0 (\nabla_\mu \omega_t(x))^a + g_0^2 w_{x,\mu} \partial^a_{x,\mu} \big] P_t(U). \quad (2.20)$$

In this case the equilibrium distribution cannot be expressed as an exponential of an action, except at the lowest order of perturbation theory. This distribution can, however, be worked out order by order in perturbation theory following the lines of [11,9]. In this section we focus on the lowest order result in the coupling. For simplicity we take for S_G the Wilson-plaquette action. The results of the section trivially extend to the improved actions considered in [6].

At this order in perturbation theory the Langevin equation (2.13) can be written as [6],

$$\begin{aligned} \partial_t A_\mu(t,x) = \{ \partial^*_\rho \partial_\rho \delta_{\mu\nu} + (\lambda_0 - 1) \partial_\mu \partial^*_\nu \} A_\nu(t,x) \\ - \delta_{\mu 0} \frac{\lambda_0}{T^3 L^3} \sum_{y_0=0}^{T-1} \sum_{\mathbf{y}} A_0(t,y) + \eta_\mu(t,x) + \mathrm{O}(g_0), \end{aligned} \quad (2.21)$$

where the second term on the right hand side of the equation is absent in the case of open-SF boundary conditions. Note that the action of the derivatives near the boundaries of the lattice is defined by extending the fields beyond the range $0 \le x_0 \le T$, and by using their Fourier representation to define the values of the fields outside this range [6].

Having this noticed, the FP equation corresponding to (2.21) reads,

$$\begin{aligned} \partial_t P_t(A) = \sum_{x,\mu,a} \frac{\delta}{\delta A^a_\mu(x)} \Big\{ - \partial^*_\nu F^a_{\nu\mu} - \lambda_0 \partial_\mu \partial^*_\nu A^a_\nu(x) \\ + \delta_{\mu 0} \frac{\lambda_0}{T^3 L^3} \sum_{y_0=0}^{T-1} \sum_{\mathbf{y}} A^a_0(y) + w_{x,\mu} \frac{\delta}{\delta A^a_\mu(x)} \Big\} P_t(A), \end{aligned} \quad (2.22)$$

where $\delta/\delta A^a_\mu(x)$ stands for the derivative with respect to the $A^a_\mu(x)$ component of the gauge potential, while the field strength tensor is defined as,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad (2.23)$$

with ∂_μ being the forward-lattice derivative.

The above equation can be recast into the form,

$$\partial_t P_t(A) = \sum_{x,\mu,a} w_{x,\mu} \frac{\delta}{\delta A^a_\mu(x)} \Big\{ \frac{\delta S_G(A)}{\delta A^a_\mu(x)} + \frac{\delta S_{\mathrm{gf}}(A)}{\delta A^a_\mu(x)} + \frac{\delta}{\delta A^a_\mu(x)} \Big\} P_t(A), \quad (2.24)$$

where for both SF and open-SF boundary conditions $S_G(A)$ is the gauge action given by [6],

$$S_G(A) = \tfrac{1}{2} \sum_{\mathbf{x}} \Big\{ \sum_{x_0=0}^{T-1} [F^a_{0k}(x)]^2 + {\sum_{x_0=0}^{T}}' \tfrac{1}{2} [F^a_{kl}(x)]^2 \Big\}, \quad (2.25)$$

where the primed summation symbol indicates that the terms at $x_0 = 0$ and $x_0 = T$ are given a weight of 1/2.

The second contribution on the right hand side of (2.24) is expressed, instead, in terms of the gauge fixing action,

$$S_{\rm gf}(A) = \tfrac{1}{2}\lambda_0(\eth A, \eth A). \tag{2.26}$$

The definition of the scalar product $(\cdot,\cdot)$, and of the operator $\eth : \mathfrak{su}(N) \to \mathfrak{su}(N)$ depends on the boundary conditions. We refer to Section 4 of [6] for the details.

In conclusion, given our choice of gauge damping function, at the lowest order in the coupling the equilibrium theory for the modified Langevin equation (2.13) corresponds to the gauge fixed theory defined in [6], i.e.,

$$P(A) \propto e^{-S(A)} \qquad \text{with} \qquad S = S_G + S_{\rm gf}. \tag{2.27}$$

Equivalently, this can be seen by a direct computation of the basic two-point functions of the gauge potential using the Langevin equation (see Appendix A).

3 Discrete integrators

As presented in the previous subsections, the Langevin equation that needs to be solved is,

$$\partial_t U_t(x,\mu) = [-F_{x,\mu}(U_t) + g_0\eta_t(x,\mu)]U_t(x,\mu), \tag{3.1}$$

where for later convenience we introduced the *drift force*,

$$F_{x,\mu}(U_t) = g_0^2 w_{x,\mu}(\partial^a_{x,\mu} S_G)(U_t)T^a + g_0\nabla_\mu\omega_t(x). \tag{3.2}$$

This equation can be solved by first discretizing the stochastic time as $t = n\epsilon$, $n \in \mathbb{N}$, where ϵ is the step-size. Then, a given integration scheme is chosen such that the correct equation is recovered for $\epsilon \to 0$. In the following we present two possible integration schemes which satisfy this requirement.

3.1 Euler scheme

Given some initial field configuration $U_{t=0}(x,\mu)$, the Euler scheme is defined by the iteration of the elementary step,

$$U_{t+\epsilon}(x,\mu) = e^{-f_t(x,\mu)}U_t(x,\mu), \tag{3.3}$$

where the force field $f_t(x,\mu) \in \mathfrak{su}(N)$ is defined as,

$$f_t(x,\mu) = \epsilon F_{x,\mu}(U_t) + \sqrt{\epsilon}\, g_0\eta_t(x,\mu), \tag{3.4}$$

and the noise field is normalized such as,

$$\langle \eta^a_t(x,\mu)\eta^b_s(y,\nu)\rangle = 2w_{x,\mu}\delta^{ab}\delta_{\mu\nu}\delta_{ts}\delta_{xy}. \tag{3.5}$$

(Note in particular that the sign of the noise in (3.4) can be chosen at wish.) It is easy to show that this integration scheme introduces errors of $\mathrm{O}(\epsilon)$ in expectation values of generic observables. These need to be extrapolated away by taking the limit $\epsilon \to 0$ [12,13].

3.2 Runge-Kutta scheme

Runge-Kutta schemes allow to reduce discretization errors in the integration of the Langevin equation. On the other hand, so far, only second order integrators have been derived for Yang-Mills theories [12,13,14]. Through these integrators the Langevin equation is solved up to $\mathrm{O}(\epsilon^2)$ corrections. Specifically, the algorithm we consider is the one proposed in [14], which in the present case is defined by the elementary step,

$$U_{t+\epsilon}(x,\mu) = e^{-\widetilde{f}_t(x,\mu)} U_t(x,\mu), \tag{3.6}$$

with,

$$\widetilde{f}_t(x,\mu) = \left(1 + C_A \frac{5-3\sqrt{2}}{12} \epsilon g_0^2 w_{x,\mu}\right) \epsilon F_{x,\mu}(\widetilde{U}_{t+\epsilon}) + \sqrt{\epsilon} g_0 \eta_t(x,\mu). \tag{3.7}$$

Here C_A is the Casimir invariant of the adjoint representation (for $\mathrm{SU}(N)$ thus $C_A = N$). The drift force $F_{x,\mu}(\widetilde{U}_{t+\epsilon})$ is computed in terms of the tentative update,

$$\widetilde{U}_{t+\epsilon}(x,\mu) = e^{-f_t(x,\mu)} U_t(x,\mu), \tag{3.8}$$

where $f_t(x,\mu)$ is given by

$$f_t(x,\mu) = \frac{3-2\sqrt{2}}{2} \epsilon F_{x,\mu}(U_t) + \frac{2-\sqrt{2}}{2} \sqrt{\epsilon} g_0 \eta_t(x,\mu), \tag{3.9}$$

with $\eta_t(x,\mu)$ being the *same* noise field that appears in (3.7).

Remark: The inclusion of the gauge damping in the Langevin evolution can alternatively be accounted for in the following way. Let

$$\mathcal{I}_\epsilon(t) : U_t \to U_{t+\epsilon}, \tag{3.10}$$

be a single step of a discrete integrator which solves for $\epsilon \to 0$ the original Langevin equation,

$$\partial_t U_t(x,\mu) = [-g_0^2 w_{x,\mu} (\partial^a_{x,\mu} S_G)(U_t) T^a + g_0 \eta_t(x,\mu)] U_t(x,\mu). \tag{3.11}$$

This can be for example the Euler scheme (3.3) or the Runge-Kutta scheme (3.6), given the substitution,

$$F_{x,\mu}(U_t) \to F_{x,\mu}(U_t) = g_0^2 w_{x,\mu} (\partial^a_{x,\mu} S_G)(U_t) T^a, \tag{3.12}$$

and analogously for $F_{x,\mu}(\widetilde{U}_{t+\epsilon})$. We then define the gauge rotation,

$$\mathcal{G}_\epsilon(t): \quad U_t(x,\mu) \to e^{\epsilon g_0 \omega_t(x)}\, U_t(x,\mu)\, e^{-\epsilon g_0 \omega_t(x+\hat{\mu})}, \tag{3.13}$$

where $\omega_t(x)$ is given by (2.16) or (2.17) depending on whether open-SF or SF boundary conditions are considered. The complete update cycle from t to $t+\epsilon$ which implements the gauge-damped Langevin equation (2.13) is now simply obtained by the application of the integrator $\mathcal{I}_\epsilon(t)$ followed by the gauge rotation $\mathcal{G}_\epsilon(t+\epsilon)$.

A Gluon propagator in stochastic perturbation theory

A.1 Definitions

In this appendix we want to compute the basic two-point functions of the gauge potential at the lowest order in perturbation theory for the equilibrium theory corresponding to the modified Langevin equation (2.13). To this end, we recall the result for the lowest order Langevin equation when S_G is taken to be the Wilson-plaquette action [6],

$$\partial_t A_\mu(t,x) = \{\partial^*_\rho\partial_\rho\delta_{\mu\nu} + (\lambda_0 - 1)\partial_\mu\partial^*_\nu\}A_\nu(t,x) + \\ - \delta_{\mu 0}\frac{\lambda_0}{T^3L^3}\sum_{y_0=0}^{T-1}\sum_{\mathbf{y}} A_0(t,y) + \eta_\mu(t,x) + \mathrm{O}(g_0), \quad \text{(A.1)}$$

where,

$$\langle \eta^a_\mu(t,x)\rangle = 0, \qquad \langle \eta^a_\mu(t,x)\eta^b_\nu(s,y)\rangle = 2w_{x,\mu}\delta^{ab}\delta_{\mu\nu}\delta(t-s)\delta_{xy}. \quad \text{(A.2)}$$

We also remind the reader that the second term on the r.h.s. of the equation (A.1) is not present if open-SF boundary conditions are considered, and that the derivatives close to the boundaries of the lattice are defined through the Fourier representation of the fields. In particular, using the Fourier representation introduced in [6], in momentum space the Langevin equation (A.1) assumes the simple diagonal form [6],

$$\partial_t \tilde{A}_\mu(t,p) = -\{\hat{p}^2\delta_{\mu\nu} + (\lambda_0 - 1)\hat{p}_\mu\hat{p}_\nu\}\tilde{A}_\nu(t,p) - \delta_{p0}\delta_{\mu 0}\frac{\lambda_0}{T^2}\tilde{A}_0(t,0) + \tilde{\eta}_\mu(t,p). \quad \text{(A.3)}$$

Here the Fourier transform of the noise field $\eta_\mu(t,x)$ is analogously defined to the one of the gauge potential $A_\mu(t,x)$. Specifically, this means that for SF boundary conditions we have,

$$\eta_0(t,x) = \frac{2}{TL^3}{\sum_p}' \cos(p_0x_0 + \tfrac{1}{2}p_0)e^{i\mathbf{px}}\,\tilde{\eta}_0(t,p), \quad \text{(A.4)}$$

$$\eta_k(t,x) = \frac{2i}{TL^3}{\sum_p}' \sin(p_0x_0)e^{i\mathbf{px}+\frac{i}{2}p_k}\,\tilde{\eta}_k(t,p), \qquad \tilde{\eta}_k(t,p)\big|_{p_0=0} = 0, \quad \text{(A.5)}$$

where the momenta ranges in the sums and related definitions can be found in Section 3 of [6]. This implies for the momentum components the representation,

$$\tilde{\eta}_0(t,p) = \sum_{x_0=0}^{T-1}\sum_{\mathbf{x}} \cos(p_0x_0 + \tfrac{1}{2}p_0)e^{-i\mathbf{px}}\,\eta_0(t,x), \quad \text{(A.6)}$$

$$\tilde{\eta}_k(t,p) = -i\sum_{x_0=1}^{T-1}\sum_{\mathbf{x}} \sin(p_0x_0)e^{-i\mathbf{px}-\frac{i}{2}p_k}\,\eta_k(t,x), \qquad \eta_k(t,x)\big|_{x_0=0,T} = 0. \quad \text{(A.7)}$$

Given these definitions, it is easy to work out the corresponding expectation values (A.2) for the momentum components of the noise field. In particular, for the case of SF boundary conditions, the only non-vanishing two-point functions are given by,

$$\langle \tilde{\eta}^a_0(t,p)^*\tilde{\eta}^b_0(s,q)\rangle = TL^3\,\delta^{ab}\delta(t-s)\delta_{pq}(1+\delta_{p_00}), \quad \text{(A.8)}$$

$$\langle \tilde{\eta}^a_k(t,p)^*\tilde{\eta}^b_k(s,q)\rangle = TL^3\,\delta^{ab}\delta(t-s)\delta_{pq}(1-\delta_{p_00}). \quad \text{(A.9)}$$

For the case of open-SF boundary conditions the Fourier representation of the noise field is given by,

$$\eta_0(t,x) = \frac{2i}{TL^3}\sum_p \sin(p_0x_0 + \tfrac{1}{2}p_0)e^{i\mathbf{px}}\,\tilde{\eta}_0(t,p), \tag{A.10}$$

$$\eta_k(t,x) = \frac{2}{TL^3}\sum_p \cos(p_0x_0)e^{i\mathbf{px}+\frac{i}{2}p_k}\,\tilde{\eta}_k(t,p), \tag{A.11}$$

where again the momenta ranges in the sums as well as related definitions can be found in Section 3 of [6]. This implies the representation,

$$\tilde{\eta}_0(t,p) = -i\sum_{x_0=0}^{T-1}\sum_{\mathbf{x}} \sin(p_0x_0 + \tfrac{1}{2}p_0)e^{-i\mathbf{px}}\,\eta_0(t,x), \tag{A.12}$$

$$\tilde{\eta}_k(t,p) = \sum_{x_0=0}^{T}{}'\sum_{\mathbf{x}} \cos(p_0x_0)e^{-i\mathbf{px}-\frac{i}{2}p_k}\,\eta_k(t,x). \tag{A.13}$$

Given these definitions, in the case of open-SF boundary conditions the two-point functions (A.2) for the momentum components of the noise field can collectively be expressed as,

$$\langle\tilde{\eta}^a_\mu(t,p)^*\tilde{\eta}^b_\nu(s,q)\rangle = TL^3\,\delta^{ab}\delta_{\mu\nu}\delta(t-s)\delta_{pq}. \tag{A.14}$$

A.2 Basic two-point functions

For any non-zero momentum p a solution of the Langevin equation is given by,

$$\tilde{A}_\mu(t,p) = \int_0^t \left\{\frac{e^{-(t-\tau)\hat{p}^2}}{\hat{p}^2}(\hat{p}^2\delta_{\mu\nu} - \hat{p}_\mu\hat{p}_\nu) + \frac{e^{-\lambda_0(t-\tau)\hat{p}^2}}{\hat{p}^2}\hat{p}_\mu\hat{p}_\nu\right\}\tilde{\eta}_\nu(\tau,p)\,\mathrm{d}\tau, \tag{A.15}$$

where here and in the following we assume for simplicity the initial condition $\tilde{A}_\mu(0,p) = 0$. The above equation can conveniently be rewritten as,

$$\tilde{A}_\mu(t,p) = \int_0^t \left\{e^{-(t-\tau)\hat{p}^2}T_{\mu\nu}(p) + e^{-\lambda_0(t-\tau)\hat{p}^2}L_{\mu\nu}(p)\right\}\tilde{\eta}_\nu(\tau,p)\,\mathrm{d}\tau, \tag{A.16}$$

by introducing the orthogonal projectors,

$$T_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}^2}, \qquad L_{\mu\nu}(p) = \delta_{\mu\nu} - T_{\mu\nu}(p) = \frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}^2}. \tag{A.17}$$

For $p = 0$, instead (and thus for SF boundary conditions), we have that the only non-vanishing Fourier component of the field is,

$$\tilde{A}_\mu(t,0) = \delta_{\mu 0}\int_0^t e^{-\lambda_0(t-\tau)/T^2}\tilde{\eta}_0(\tau,0)\,\mathrm{d}\tau. \tag{A.18}$$

Given the general solution of the Langevin equation we can now consider the basic two-point functions of the theory. We will treat separately the case where the momentum of the gauge potentials entering the two-point functions satisfies $p_0 = 0$ or $p_0 \neq 0$.

For any $p_0 \neq 0$ (and thus in all cases with open-SF boundary conditions), a few lines of algebra show that the equal-time two-point functions of the gauge potential are given by,

$$\langle \tilde{A}^a_\mu(t,p)^* \tilde{A}^b_\nu(t,q) \rangle = \frac{1}{2} T L^3 \delta^{ab} \delta_{pq} \frac{1}{\hat{p}^2} \Big[T_{\mu\nu}(p) \big(1 - e^{-2t\hat{p}^2}\big) + \lambda_0^{-1} L_{\mu\nu}(p) \big(1 - e^{-2\lambda_0 t \hat{p}^2}\big) \Big]. \tag{A.19}$$

In the limit $t \to \infty$ one thus recovers the known results for these correlators [6,15],

$$\begin{aligned} \lim_{t\to\infty} \langle \tilde{A}^a_\mu(t,p)^* \tilde{A}^b_\nu(t,q) \rangle &= \frac{1}{2} T L^3 \delta^{ab} \delta_{pq} \frac{1}{\hat{p}^2} \Big[T_{\mu\nu}(p) + \lambda_0^{-1} L_{\mu\nu}(p) \Big], \\ &= \frac{1}{2} T L^3 \delta^{ab} \delta_{pq} \frac{1}{\hat{p}^2} \Big[\delta_{\mu\nu} + (\lambda_0^{-1} - 1) \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}^2} \Big]. \end{aligned} \tag{A.20}$$

For all momenta with $p_0 = 0$, given the relations (A.8) and (A.9), one immediately concludes that,

$$\langle \tilde{A}_\mu(t,p)^* \tilde{A}_\nu(t,q) \rangle |_{p_0=0} = 0, \quad \text{if } \mu \neq 0 \text{ or } \nu \neq 0. \tag{A.21}$$

This noticed, the case where $p_0 = 0$ but $p_k \neq 0$ for some $k = 1,2,3$, is analogous to what discussed above and one has that,

$$\lim_{t\to\infty} \langle \tilde{A}^a_0(t,p)^* \tilde{A}^b_0(t,q) \rangle \big|_{p_0=0, p\neq 0} = T L^3 \delta^{ab} \delta_{pq} \frac{1}{\hat{p}^2}. \tag{A.22}$$

For $p = 0$, instead, it is easy to show that,

$$\langle \tilde{A}^a_0(t,0)^* \tilde{A}^b_0(t,0) \rangle = T L^3 \delta^{ab} \lambda_0^{-1} T^2 \big(1 - e^{-2\lambda_0 t/T^2}\big), \tag{A.23}$$

and thus,

$$\lim_{t\to\infty} \langle \tilde{A}^a_0(t,0)^* \tilde{A}^b_0(t,0) \rangle = T L^3 \delta^{ab} \lambda_0^{-1} T^2. \tag{A.24}$$

In conclusion, we could explicitly show that at the lowest order in the coupling, the results for the equilibrium theory associated to the modified Langevin equation (2.13) correspond to those of the gauge fixed theory defined in [6].

References

[1] F. Di Renzo, G. Marchesini, P. Marenzoni, and E. Onofri, *Lattice perturbation theory on the computer*, *Nucl. Phys. Proc. Suppl.* **34** (1994) 795–798.

[2] F. Di Renzo, E. Onofri, G. Marchesini, and P. Marenzoni, *Four loop result in SU(3) lattice gauge theory by a stochastic method: Lattice correction to the condensate*, *Nucl. Phys.* **B426** (1994) 675–685.

[3] F. Di Renzo and L. Scorzato, *Numerical stochastic perturbation theory for full QCD*, *JHEP* **0410** (2004) 073.

[4] M. Brambilla, M. Dalla Brida, D. Hesse, F. Di Renzo, and S. Sint, *Numerical Stochastic Perturbation Thoery for Schrödinger Functional schemes*, *PoS* **LATTICE2013** (2013) 325.

[5] M. Dalla Brida and D. Hesse, *Numerical Stochastic Perturbation Theory and the Gradient Flow*, *PoS* **LATTICE2013** (2014) 326.

[6] M. Lüscher, *Normalization of the gradient flow coulping in LQCD*, notes (January 2014).

[7] M. Lüscher, *Yang-Mills gradient flow in stochastic perturbation theory*, notes (May 2015).

[8] J. Zinn-Justin, *Quantum field theory and critical phenomena*, *International Series of Monographs on Physics* (2002).

[9] M. Lüscher, *Fokker-Planck equation in stochastic perturbation theory*, notes (October 2015).

[10] P. H. Damgaard and H. Hüffel, *Stochastic Quantization*, *Phys. Rept.* **152** (1987) 227.

[11] E. G. Floratos, J. Iliopoulos, and D. Zwanziger, *A Covariant Ghost Free Perturbation Expansion for Yang-Mills Theories*, *Nucl. Phys.* **B241** (1984) 221.

[12] G. G. Batrouni, G. R. Katz, A. S. Kronfeld, G. P. Lepage, B. Svetitsky, and K. G. Wilson, *Langevin Simulations of Lattice Field Theories*, *Phys. Rev.* **D32** (1985) 2736.

[13] A. S. Kronfeld, *Dynamics of Langevin simulations*, *Prog. Theor. Phys. Suppl.* **111** (1993) 293–312.

[14] G. S. Bali, C. Bauer, A. Pineda, and C. Torrero, *Perturbative expansion of the energy of static sources at large orders in four-dimensional SU(3) gauge theory*, *Phys. Rev.* **D87** (2013) 094517.

[15] M. Lüscher, *Step scaling and the Yang-Mills gradient flow*, *JHEP* **06** (2014) 105.