Yang-Mills gradient flow in stochastic perturbation theory

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1. Introduction

In stochastic perturbation theory [1–5], the gradient flow equation must be solved order by order in the gauge coupling. A gauge-damping term should preferably be included in the equation in order to stabilize its numerical integration. These and other details are discussed in this note.

The lattice theory is set up as in refs. [6,7]. Both SF [8] and open-SF [9] boundary conditions are considered and the link variables are assumed to take values in $\mathrm{SL}(N,\mathbb{C})$ rather than $\mathrm{SU}(N)$, as is the case in instantaneous stochastic perturbation theory [10].

2. Flow equation

Only the simplest form of the flow equation is considered in the following, but the computational strategies discussed are more generally applicable.

2.1 Equation without gauge damping

Let S_{w} be the tree-level $\mathrm{O}(a)$-improved Wilson gauge action. The active link variables $V_t(x,\mu)$ at flow time $t \geq 0$ are determined by the boundary condition

$$V_t(x,\mu)|_{t=0} = U(x,\mu) \tag{2.1}$$

and the flow equation

$$\partial_t V_t(x,\mu) V_t(x,\mu)^{-1} = -w_{x,\mu} g_0^2 \{\partial^a_{x,\mu} S_{\mathrm{w}}(V_t)\} T^a, \qquad 0 \leq x_0 < T, \tag{2.2}$$

where $\partial^a_{x,\mu} S_{\rm w}(U)$ denotes the partial derivative of the action with respect to the link variable $U(x,\mu)$ in direction of the SU(N) generator T^a.

The weight factor $w_{x,\mu}$ in eq. (2.2) depends on the chosen boundary conditions. For SF boundary conditions

$$w_{x,\mu} = \begin{cases} 0 & \text{if } x_0 = 0 \text{ and } \mu > 0, \\ 1 & \text{otherwise,} \end{cases} \tag{2.3}$$

while for open-SF boundary conditions

$$w_{x,\mu} = \begin{cases} 2 & \text{if } x_0 = 0 \text{ and } \mu > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.4}$$

Assigning weight 2 to the spatial links at time 0 is required in this case for O(a) improvement [11].

2.2 Gauge damping

As explained in ref. [6], the solution $V_t(x,\mu)$ of the modified flow equation

$$\partial_t V_t(x,\mu) V_t(x,\mu)^{-1} = -w_{x,\mu} g_0^2 \{\partial^a_{x,\mu} S_{\rm w}(V_t)\} T^a + D_\mu \omega_t(x), \tag{2.5}$$

$$D_\mu \omega_t(x) = V_t(x,\mu)\omega_t(x+\hat\mu)V_t(x,\mu)^{-1} - \omega_t(x), \tag{2.6}$$

coincides with the solution of eq. (2.2) up to a time dependent gauge transformation, for any field $\omega_t(x)$ with values in $\mathfrak{sl}(N,\mathbb{C})$ satisfying

$$\omega_t(x)|_{x_0=T} = 0 \tag{2.7}$$

and additionally

$$\partial_k \omega_t(x)|_{x_0=0} = 0, \quad k = 1,2,3, \tag{2.8}$$

if SF boundary conditions are chosen.

A damping of the gauge modes can be achieved in this way by setting

$$\omega_t(x) = \sum_{\mu=0}^{3} \partial^*_\mu C_t(x,\mu) \tag{2.9}$$

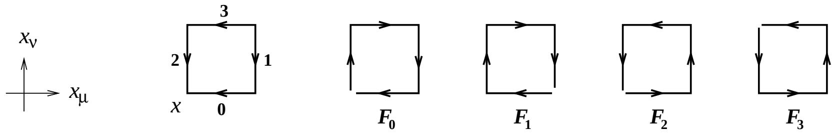

Fig. 1. Labeling of the link variables at the edges of the plaquette in the (μ,ν)-plane with lower-left corner x. The four Wilson lines on the right represent the products of link variables contributing to the associated force components.

and

$$\omega_t(x) = (1 - x_0/T)\frac{1}{T^2L^3}\sum_y C_t(y,0) + \sum_{\mu=0}^{3}\partial_\mu^* C_t(x,\mu) \tag{2.10}$$

for open-SF and SF boundary conditions, respectively. In these equations,

$$C_t(x,\mu) = \frac{1}{2}\left\{V_t(x,\mu) - V_t(x,\mu)^{-1}\right\} - \frac{1}{2N}\mathrm{tr}\left\{V_t(x,\mu) - V_t(x,\mu)^{-1}\right\} \tag{2.11}$$

takes values in $\mathfrak{sl}(N,\mathbb{C})$ and

$$\partial_0^* C_t(x,0)|_{x_0=0} = \begin{cases} 2C_t(x,0) & \text{for open-SF bc,} \\ 0 & \text{for SF bc,} \end{cases} \tag{2.12}$$

by convention (see sect. 3 for further explanations).

2.3 Force field

The first term in the force field

$$F_t(x,\mu) = -w_{x,\mu} g_0^2 \{\partial_{x,\mu}^a S_{\mathrm{w}}(V_t)\} T^a + D_\mu \omega_t(x) \tag{2.13}$$

can be evaluated by running through all plaquettes on the lattice and accumulating the contributions to the force field at the edges of the plaquettes. The contribution to $F_t(x,\mu)$ of the plaquette p shown in fig. 1, for example, is equal to

$$-\frac{1}{2}w_{x,\mu}w_0(p)\left\{V - V^{-1} - \frac{1}{N}\mathrm{tr}\left[V - V^{-1}\right]\right\}, \tag{2.14}$$

where

$$V = V_t(x,\mu)V_t(x+\hat{\mu},\nu)V_t(x+\hat{\nu},\mu)^{-1}V_t(x,\nu)^{-1} \tag{2.15}$$

is the Wilson loop labeled F_0 in fig. 1 and $w_0(p)$ the weight of the plaquette in the Wilson action.

The gauge damping term in the force field can be computed following the lines of the previous subsection. First the gauge potential $C_t(x,\mu)$ is calculated on all links, then the field $\omega_t(x)$ and finally the gauge-covariant forward difference of that field.

2.4 3rd order Runge–Kutta integration

It is helpful to write the flow equation (2.5) in an abstract form

$$\partial_t V_t = Z(V_t)V_t, \tag{2.16}$$

where the gauge field V_t at flow time t is considered to be an element of a (high-dimensional) Lie group and the force field $Z(V)$ an element of the associated Lie algebra. The integration of eq. (2.16) proceeds in time steps of size ϵ. Assuming V_t is known, the fields

$$W_0 = V_t, \tag{2.17}$$

$$W_1 = \exp\{\tfrac{1}{4}Z_0\}W_0, \tag{2.18}$$

$$W_2 = \exp\{\tfrac{8}{9}Z_1 - \tfrac{17}{36}Z_0\}W_1, \tag{2.19}$$

$$W_3 = \exp\{\tfrac{3}{4}Z_2 - \tfrac{8}{9}Z_1 + \tfrac{17}{36}Z_0\}W_2, \tag{2.20}$$

are computed, where

$$Z_i = \epsilon Z(W_i), \qquad i = 0,1,2. \tag{2.21}$$

The last field, W_3, can be shown to coincide with $V_{t+\epsilon}$ up to an error of order ϵ^4.

3. Perturbation expansion

3.1 Stochastic fields

In numerical stochastic perturbation theory, the basic link variables are expanded in powers of the coupling g_0,

$$U(x,\mu) = 1 + g_0U_1(x,\mu) + g_0^2U_2(x,\mu) + \ldots + g_0^nU_n(x,\mu), \tag{3.1}$$

up to the desired order n. The coefficients $U_k(x,\mu)$ are complex $N\times N$ matrices satisfying some algebraic relations so as to ensure that $U(x,\mu)$ is in SU(N) (or SL($N,\mathbb{C}$) in the case of ISPT) up to terms of order g_0^{n+1} (see appendix A).

3.2 Flow equation in perturbation theory

The time-dependent field $V_t(x,\mu)$ and the force field $F_t(x,\mu)$ can be expanded in powers of the coupling too and the flow equation then turns into a system of equations for the coefficients $V_{t,k}(x,\mu)$ of the time-dependent field. At any given order $k\geq 1$ in the coupling, the equation to be solved is of the form

$$\partial_t V_{t,k}(x,\mu)=\sum_{\nu=0}^{3}\partial_\nu^*\partial_\nu V_{t,k}(x,\mu)-\delta_{\mu 0}\frac{1}{T^3L^3}\sum_y V_{t,k}(y,0)$$
$$+\text{ non-linear terms,} \tag{3.2}$$

where the non-linear terms depend on the components $V_{t,j}$ with $j<k$ only and the second term on the right is absent when open-SF boundary conditions are chosen. Near the boundaries of the lattice, the boundary conditions deriving from the flow equation are

$$\partial_0^* V_{t,k}(x,0)\big|_{x_0=0}=V_{t,k}(x,j)\big|_{x_0=0}=0,\qquad j=1,2,3, \tag{3.3}$$

for SF boundary conditions,

$$(V_{t,k}(x,0)+V_{t,k}(x-\hat{0},0))\big|_{x_0=0}=\partial_0(V_{t,k}(x,j)+V_{t,k}(x-\hat{0},j))\big|_{x_0=0}=0 \tag{3.4}$$

for open-SF boundary conditions and

$$\partial_0^* V_{t,k}(x,0)\big|_{x_0=T}=V_{t,k}(x,j)\big|_{x_0=T}=0 \tag{3.5}$$

for both.

Equation (3.2) implies that the gradient flow tends to smooth field components, including the gauge modes, and eventually drives all of them to zero (cf. ref. [6]).

3.3 Numerical integration

The system of gradient flow equations can be solved along the lines of subsect. 2.4. To this end, the force field, the exponential functions in eqs. (2.18)–(2.20) and the fields W_i, $i=0,\ldots,3$ are all expanded in powers of the coupling. The integration rule then allows the field components to be computed recursively, from step to step and from order 0 to order n in each step.

Appendix A

Let

$$U = 1 + g_0 U_1 + g_0^2 U_2 + \ldots + g_0^n U_n \tag{A.1}$$

be a polynomial in the gauge coupling with complex $N \times N$ matrix coefficients U_k. For U to be unitary and unimodular up to terms of order g_0^{n+1} the conditions are

$$U_k + U_k^\dagger + \sum_{j=1}^{k-1} U_j U_{k-j}^\dagger = 0, \tag{A.2}$$

$$\mathrm{tr}\,\{L_k\} = 0, \quad k = 1, \ldots, n, \tag{A.3}$$

respectively, where the matrices L_k are defined through the expansion

$$\ln U = g_0 L_1 + g_0^2 L_2 + \ldots + g_0^n L_n + \mathrm{O}(g_0^{n+1}). \tag{A.4}$$

If U is nearly unitary, i.e. if the elements of the matrices

$$E_k = U_k + U_k^\dagger + \sum_{j=1}^{k-1} U_j U_{k-j}^\dagger \tag{A.5}$$

are orders of magnitude smaller than 1, the matrix can be reunitarized by replacing U_k by the kth coefficient in the expansion of $(1 - \frac{1}{2}E)U$. And if

$$e_k = \mathrm{tr}\{L_k\} \tag{A.6}$$

does not vanish, but is much smaller than 1, the determinant of U can be reset to unity by multiplication with the (complex) factor $1 - e/N$.

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