# O(a) boundary effects on the gradient-flow coupling

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## 1. Introduction

The present note extends the previous notes on the normalization of the gradient-flow coupling in lattice gauge theory [1]. Familiarity with ref. [1] is assumed and the notation used there is taken over completely. Equations in [1] are referred to by the equation number with prefix I such as (I.3.17).

The gradient-flow coupling receives contributions of $\mathrm{O}(a)$ from the boundaries of the lattice in the time direction. These can be canceled by adjusting the improvement coefficients $c_{\mathrm{G}}$ and $c'_{\mathrm{G}}$ (see sect. 2 of [1]). In this note, the effect of the associated boundary counterterms on the gradient-flow coupling is computed to leading order in the gauge coupling.

## 2. Expansion of the boundary action

Like the bulk action, the boundary terms in the action proportional to

$$\delta c_{\mathrm{G}} = c_{\mathrm{G}} - 1 \quad \text{and} \quad \delta c'_{\mathrm{G}} = c'_{\mathrm{G}} - 1 \tag{2.1}$$

can be expanded in powers of the gauge coupling. Both coefficients (2.1) are of order $g_0^2$. In this section, the boundary terms are worked out to leading order.

*2.1 SF boundary conditions*

In this case, the boundary terms are

$$\delta S_{\mathrm{G}} = \frac{1}{g_0^2} \delta c_{\mathrm{G}} c_0 \sum_{\mathcal{C} \in \mathcal{S}_{0,t} \cup \mathcal{S}'_{0,t}} \mathrm{tr}\{1 - U(\mathcal{C})\}, \tag{2.2}$$

where $\mathcal{S}_{0,t}$ and $\mathcal{S}'_{0,t}$ are the sets of time-like oriented plaquettes that touch the boundaries at time 0 and $T$, respectively. To leading order, this expression evaluates to

$$\delta S_{\mathrm{G}} = \tfrac{1}{2} \delta c_{\mathrm{G}} c_0 \sum_{\boldsymbol{x}} \Big\{ \big[F_{0k}^a(x)\big]^2_{x_0=T-1} + \big[F_{0k}^a(x)\big]^2_{x_0=0} \Big\} + \mathrm{O}(g_0^3) \tag{2.3}$$

[cf. eq. (I.3.22)].

*2.2 Open-SF boundary conditions*

With these boundary conditions, the boundary counterterms near time 0 and near time $T$ do not have the same form. Denoting the sets of oriented spatial plaquette and double-plaquette loops at time 0 by $\mathcal{S}_{0,s}$ and $\mathcal{S}_{1,s}$, the counterterms are given by

$$\delta S_{\mathrm{G}} = \frac{1}{g_0^2} \Bigg\{ \delta c'_{\mathrm{G}} c_0 \sum_{\mathcal{C} \in \mathcal{S}'_{0,t}} \mathrm{tr}\{1 - U(\mathcal{C})\}$$

$$+ \tfrac{1}{2} \delta c_{\mathrm{G}} \sum_{k=0}^{1} \sum_{\mathcal{C} \in \mathcal{S}_{k,s}} c_k \mathrm{tr}\{1 - U(\mathcal{C})\} \Bigg\}. \tag{2.4}$$

In terms of the gauge potential, the expression

$$\delta S_{\mathrm{G}} = \sum_{\boldsymbol{x}} \Bigg\{ \tfrac{1}{2} \delta c'_{\mathrm{G}} c_0 \big[F_{0k}^a(x)\big]^2_{x_0=T-1} + \tfrac{1}{8} \delta c_{\mathrm{G}} \big[F_{kl}^a(x)\big]^2_{x_0=0}$$

$$- \tfrac{1}{4} \delta c_{\mathrm{G}} c_1 \big[\partial_k^* F_{kl}^a(x)\big]^2_{x_0=0} \Bigg\} + \mathrm{O}(g_0^3) \tag{2.5}$$

is then obtained (the indices $a, k$ and $l$ are to be summed over).

## 3. Correlation functions of the field tensor

At leading order in the coupling, the O($a$) boundary effects on the expectation value of the density $E(t,x)$ are given by a sum of terms like

$$\sum_{\boldsymbol{y}} \left\{ \langle G^a_{\mu\nu}(t,x) F^b_{0k}(y) \rangle \right\}^2 , \tag{3.1}$$

where the expectation value is to be computed in the free-field limit. The Fourier representation of the two-point functions in these formulae can be written down explicitly. They all involve the combination

$$\mathcal{K}_{\mu\nu\rho\sigma}(p) = \hat{p}_\mu \hat{p}_\rho D_{\nu\sigma}(p) - \hat{p}_\mu \hat{p}_\sigma D_{\nu\rho}(p) + \hat{p}_\nu \hat{p}_\sigma D_{\mu\rho}(p) - \hat{p}_\nu \hat{p}_\rho D_{\mu\sigma}(p) \tag{3.2}$$

of the gauge propagator (see sect. 5 in [1]). The tensor (3.2) is anti-symmetric in the indices $\mu, \nu$ and $\rho, \sigma$ and symmetric under interchanges of these pairs of indices.

*3.1 SF boundary conditions*

In this case, the four-momenta $p$ range over the sets (I.3.3) and (I.3.4). The field-tensor correlation functions that contribute to the O($a$) boundary effects are

$$\begin{aligned} \langle G^a_{0k}(t,x) F^b_{0j}(y) \rangle = \delta^{ab} \frac{2g_0}{TL^3} {\sum_p}' \cos(p_0 x_0) \cos(p_0 y_0 + \tfrac{1}{2} p_0) \mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y}) - \frac{i}{2} p_j} \\ \times \mathrm{e}^{-t\hat{p}^2} \cos(\tfrac{1}{2} p_0) \cos(\tfrac{1}{2} p_k) \mathcal{K}_{0k0j}(p), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \langle G^a_{kl}(t,x) F^b_{0j}(y) \rangle = \delta^{ab} \frac{2ig_0}{TL^3} {\sum_p}' \sin(p_0 x_0) \cos(p_0 y_0 + \tfrac{1}{2} p_0) \mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y}) - \frac{i}{2} p_j} \\ \times \mathrm{e}^{-t\hat{p}^2} \cos(\tfrac{1}{2} p_k) \cos(\tfrac{1}{2} p_l) \mathcal{K}_{kl0j}(p). \end{aligned} \tag{3.4}$$

Under time reflections

$$x_0 \to T - x_0, \quad y_0 \to T - 1 - y_0, \tag{3.5}$$

the first (second) of these correlation functions is symmetric (anti-symmetric).

*3.2 Open-SF boundary conditions*

The time components of the momenta run over the set (I.3.12) in this case and the relevant two-point functions are

$$\langle G^a_{0k}(t,x)F^b_{0j}(y)\rangle = \delta^{ab}\frac{2g_0}{TL^3}\sum_p \sin(p_0x_0)\sin(p_0y_0+\tfrac{1}{2}p_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}p_j} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\mathcal{K}_{0k0j}(p), \tag{3.6}$$

$$\langle G^a_{0k}(t,x)F^b_{hj}(y)\rangle = \delta^{ab}\frac{2ig_0}{TL^3}\sum_p \sin(p_0x_0)\cos(p_0y_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}(p_h+p_j)} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\mathcal{K}_{0khj}(p), \tag{3.7}$$

$$\langle G^a_{0k}(t,x)\partial^*_hF^b_{hj}(y)\rangle = \delta^{ab}\frac{2g_0}{TL^3}\sum_p \sin(p_0x_0)\cos(p_0y_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}p_j} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\hat{p}_h\mathcal{K}_{0khj}(p), \tag{3.8}$$

$$\langle G^a_{kl}(t,x)F^b_{0j}(y)\rangle = -\delta^{ab}\frac{2ig_0}{TL^3}\sum_p \cos(p_0x_0)\sin(p_0y_0+\tfrac{1}{2}p_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}p_j} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\mathcal{K}_{kl0j}(p), \tag{3.9}$$

$$\langle G^a_{kl}(t,x)F^b_{hj}(y)\rangle = \delta^{ab}\frac{2g_0}{TL^3}\sum_p \cos(p_0x_0)\cos(p_0y_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}(p_h+p_j)} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\mathcal{K}_{klhj}(p), \tag{3.10}$$

$$\langle G^a_{kl}(t,x)\partial^*_hF^b_{hj}(y)\rangle = -\delta^{ab}\frac{2ig_0}{TL^3}\sum_p \cos(p_0x_0)\cos(p_0y_0)\mathrm{e}^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-\frac{i}{2}p_j} \\ \times \mathrm{e}^{-t\hat{p}^2}\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\hat{p}_h\mathcal{K}_{klhj}(p). \tag{3.11}$$

## 4. Leading-order formula

In the following, the time $x_0$ in the definition (I.7.1) of the runnung coupling is set to $T/2$. The leading-order effect of the $\mathrm{O}(a)$ boundary counterterms on the running coupling is then

$$\delta\bar{g}^2 = -k\left\{t^2\langle E(t,x)\delta S_{\mathrm{G}}\rangle_{\mathrm{con}}\right\}_{\sqrt{8t}=cL} + \mathrm{O}(g_0^6). \tag{4.1}$$

To the order stated, the correlation function on the right of this equation is equal to a sum of squares of the type (3.1) summed over the spatial coordinates of $y$. All these coincide with the square norm of certain functions in 3-dimensional momentum space.

*4.1 SF boundary conditions*

In view of the symmetry properties of the two-point functions (3.3),(3.4), the two terms in the boundary action (2.3) contribute equally to the correlation function

$$t^2\langle E(t,x)\delta S_{\mathrm{G}}\rangle_{\mathrm{con}} = g_0^2\delta c_{\mathrm{G}} r + \mathrm{O}(g_0^6) \tag{4.2}$$

$$r = (N^2-1)\frac{2c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{\mu,\nu=0}^{3}\sum_{j=1}^{3} f_{\mu\nu j}(\boldsymbol{p})^2. \tag{4.3}$$

The functions $f_{\mu\nu j}$ are anti-symmetric under exchanges of the indices $\mu,\nu$ and explicitly given by

$$f_{0kj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0}{}' \mathrm{e}^{-t\hat{p}^2}\cos(p_0x_0)\cos(\tfrac{1}{2}p_0)^2\cos(\tfrac{1}{2}p_k)\mathcal{K}_{0k0j}(p), \tag{4.4}$$

$$f_{klj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0}{}' \mathrm{e}^{-t\hat{p}^2}\sin(p_0x_0)\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\mathcal{K}_{kl0j}(p). \tag{4.5}$$

These expressions allow the coefficient $r$ and thus $\delta\bar{g}^2$ to be computed numerically with an effort growing proportionally to the lattice volume.

*4.2 open-SF boundary conditions*

In this case, there are three distinct boundary terms contributing to the correlation function

$$t^2\langle E(t,x)\delta S_{\mathrm{G}}\rangle_{\mathrm{con}} = g_0^2\delta c_{\mathrm{G}} r + g_0^2\delta c'_{\mathrm{G}} r' + \mathrm{O}(g_0^6), \tag{4.6}$$

$$r = (N^2-1)\frac{t^2}{4L^3}\sum_{\boldsymbol{p}}\sum_{\mu,\nu=0}^{3}\sum_{h,j=1}^{3}\left\{f_{\mu\nu hj}(\boldsymbol{p})^2 - 2c_1 g_{\mu\nu hj}(\boldsymbol{p})^2\right\}, \tag{4.7}$$

$$r' = (N^2-1)\frac{c_0 t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{\mu,\nu=0}^{3}\sum_{j=1}^{3} h_{\mu\nu j}(\boldsymbol{p})^2. \tag{4.8}$$

The functions in these formulae are all anti-symmetric in the indices $\mu,\nu$ and are given by

$$f_{0khj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\sin(p_0x_0)\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\mathcal{K}_{0khj}(p), \tag{4.9}$$

$$f_{klhj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\cos(p_0x_0)\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\mathcal{K}_{klhj}(p), \tag{4.10}$$

$$g_{0khj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\sin(p_0x_0)\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\hat{p}_h\mathcal{K}_{0khj}(p), \tag{4.11}$$

$$g_{klhj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\cos(p_0x_0)\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\hat{p}_h\mathcal{K}_{klhj}(p), \tag{4.12}$$

$$h_{0kj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\cos(p_0x_0)\cos(\tfrac{1}{2}p_0)^2\cos(\tfrac{1}{2}p_k)\mathcal{K}_{0k0j}(p), \tag{4.13}$$

$$h_{klj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2}\sin(p_0x_0)\cos(\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_k)\cos(\tfrac{1}{2}p_l)\mathcal{K}_{kl0j}(p). \tag{4.14}$$

Here again, the computation of the coefficients $r$ and $r'$ requires an only moderate effort.

## 5. Other running couplings

In the case of the alternative definitions of the running coupling considered in sect. 8 of ref. [1], the O($a$) boundary effects are again of the form (4.2),(4.6), with $r$ and $r'$ replaced by $r_{\rm cs},..,r_{\rm pt}$ and $r'_{\rm cs},..,r'_{\rm pt}$, respectively.

*5.1 Computation of* $r_{\rm cs}, r_{\rm ct}, r'_{\rm cs}$ *and* $r'_{\rm ct}$

These coefficients are given by eqs. (4.3),(4.7) and (4.8) apart from the fact that the sums over the indices $\mu,\nu$ are restricted to the space- or the time-like planes. Explicitly,

$$r_{\rm cs} = (N^2-1)\frac{2c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{j=1}^{3} f_{klj}(\boldsymbol{p})^2, \tag{5.1}$$

$$r_{\rm ct} = (N^2-1)\frac{4c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{j=1}^{3} f_{0kj}(\boldsymbol{p})^2, \tag{5.2}$$

in the case of SF boundary conditions, while

$$r_{\rm cs} = (N^2-1)\frac{t^2}{4L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{h,j=1}^{3} \left\{f_{klhj}(\boldsymbol{p})^2 - 2c_1 g_{klhj}(\boldsymbol{p})^2\right\}, \tag{5.3}$$

$$r_{\rm ct} = (N^2-1)\frac{t^2}{2L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{h,j=1}^{3} \left\{f_{0khj}(\boldsymbol{p})^2 - 2c_1 g_{0khj}(\boldsymbol{p})^2\right\}, \tag{5.4}$$

$$r'_{\rm cs} = (N^2-1)\frac{c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{j=1}^{3} h_{klj}(\boldsymbol{p})^2, \tag{5.5}$$

$$r'_{\rm ct} = (N^2-1)\frac{2c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{j=1}^{3} h_{0kj}(\boldsymbol{p})^2, \tag{5.6}$$

when open-SF boundary conditions are chosen.

*5.1 Computation of* $r_{\rm ps}, r_{\rm pt}, r'_{\rm ps}$ *and* $r'_{\rm pt}$

The plaquette expressions (I.8.7),(I.8.8) for the Yang-Mills action density require a set of further momentum sums to be introduced. With SF boundary conditions,

$$r_{\rm ps} = (N^2-1)\frac{2c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{j=1}^{3}\bar{f}_{klj}(\boldsymbol{p})^2, \tag{5.7}$$

$$r_{\rm pt} = (N^2-1)\frac{2c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{j=1}^{3}\left\{\bar{f}^{+}_{kj}(\boldsymbol{p})^2+\bar{f}^{-}_{kj}(\boldsymbol{p})^2\right\}, \tag{5.8}$$

and the new momentum sums are

$$\bar{f}^{\pm}_{kj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0}{}' \mathrm{e}^{-t\hat{p}^2}\cos(p_0x_0\pm\tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_0)\mathcal{K}_{0k0j}(p), \tag{5.9}$$

$$\bar{f}_{klj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0}{}' \mathrm{e}^{-t\hat{p}^2}\sin(p_0x_0)\cos(\tfrac{1}{2}p_0)\mathcal{K}_{kl0j}(p). \tag{5.10}$$

If open-SF boundary conditions are chosen, the coefficients are given by

$$r_{\rm ps} = (N^2-1)\frac{t^2}{4L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{h,j=1}^{3}\left\{\bar{f}_{klhj}(\boldsymbol{p})^2-2c_1\bar{g}_{klhj}(\boldsymbol{p})^2\right\}, \tag{5.11}$$

$$\begin{aligned} r_{\rm pt} = (N^2-1)\frac{t^2}{4L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{h,j=1}^{3}\big\{&\bar{f}^{+}_{khj}(\boldsymbol{p})^2+\bar{f}^{-}_{khj}(\boldsymbol{p})^2 \\ &-2c_1\big(\bar{g}^{+}_{khj}(\boldsymbol{p})^2+\bar{g}^{-}_{khj}(\boldsymbol{p})^2\big)\big\}, \end{aligned} \tag{5.12}$$

$$r'_{\rm ps} = (N^2-1)\frac{c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k,l=1}^{3}\sum_{j=1}^{3}\bar{h}_{klj}(\boldsymbol{p})^2, \tag{5.13}$$

$$r'_{\rm pt} = (N^2-1)\frac{c_0t^2}{L^3}\sum_{\boldsymbol{p}}\sum_{k=1}^{3}\sum_{j=1}^{3}\left\{\bar{h}^{+}_{kj}(\boldsymbol{p})^2+\bar{h}^{-}_{kj}(\boldsymbol{p})^2\right\}, \tag{5.14}$$

where

$$\bar{f}^{\pm}_{khj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \sin(p_0 x_0 \pm \tfrac{1}{2}p_0)\mathcal{K}_{0khj}(p), \tag{5.15}$$

$$\bar{f}_{klhj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \cos(p_0 x_0)\mathcal{K}_{klhj}(p), \tag{5.16}$$

$$\bar{g}^{\pm}_{khj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \sin(p_0 x_0 \pm \tfrac{1}{2}p_0)\hat{p}_h\mathcal{K}_{0khj}(p), \tag{5.17}$$

$$\bar{g}_{klhj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \cos(p_0 x_0)\hat{p}_h\mathcal{K}_{klhj}(p), \tag{5.18}$$

$$\bar{h}^{\pm}_{kj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \cos(p_0 x_0 \pm \tfrac{1}{2}p_0)\cos(\tfrac{1}{2}p_0)\mathcal{K}_{0k0j}(p), \tag{5.19}$$

$$\bar{h}_{klj}(\boldsymbol{p}) = \frac{1}{T}\sum_{p_0} \mathrm{e}^{-t\hat{p}^2} \sin(p_0 x_0)\cos(\tfrac{1}{2}p_0)\mathcal{K}_{kl0j}(p). \tag{5.20}$$

## Appendix A

The figures shown in tables 1–6 were obtained for $N = 3$, $T = L$, $c = 0.3$ and $c_1 = 0$. Apart from the coefficients $r, r'$, the products of factors contributing to the coefficients of $-\delta c_{\mathrm{G}}$ and $-\delta c'_{\mathrm{G}}$ in the expansion of the coupling $\alpha = \bar{g}^2/4\pi$ in powers of $\alpha_0 = g_0^2/4\pi$ are listed. Comparing SF with open-SF boundary conditions, the dependence of the coupling on the improvement coefficients is thus significantly stronger in the case of SF boundary conditions.

## References

[1] M. Lüscher, *Normalization of the gradient-flow coupling in LQCD*, notes, January 2014 (revised January 2016).

Table 1. Values of $r_{\mathrm{c}}, r_{\mathrm{cs}}, r_{\mathrm{ct}}$ (Wilson action, SF bc)

| $L$ | $r_{\mathrm{c}} \times 10^4$ | $r_{\mathrm{cs}} \times 10^4$ | $r_{\mathrm{ct}} \times 10^4$ | $4\pi k_{\mathrm{c}} r_{\mathrm{c}}$ | $4\pi k_{\mathrm{cs}} r_{\mathrm{cs}}$ | $4\pi k_{\mathrm{ct}} r_{\mathrm{ct}}$ |
|---|---|---|---|---|---|---|
| 8 | 6.1276 | 1.5143 | 4.6133 | 0.4823 | 0.2413 | 0.7175 |
| 12 | 4.1986 | 0.9924 | 3.2062 | 0.3153 | 0.1507 | 0.4763 |
| 16 | 3.1775 | 0.7370 | 2.4405 | 0.2351 | 0.1102 | 0.3573 |
| 24 | 2.1314 | 0.4872 | 1.6442 | 0.1560 | 0.0721 | 0.2383 |
| 32 | 1.6020 | 0.3642 | 1.2377 | 0.1169 | 0.0537 | 0.1787 |
| 48 | 1.0696 | 0.2422 | 0.8273 | 0.0778 | 0.0356 | 0.1192 |

Table 2. Values of $r_{\mathrm{c}}, r_{\mathrm{cs}}, r_{\mathrm{ct}}$ (Wilson action, open-SF bc)

| $L$ | $r_{\mathrm{c}} \times 10^4$ | $r_{\mathrm{cs}} \times 10^4$ | $r_{\mathrm{ct}} \times 10^4$ | $4\pi k_{\mathrm{c}} r_{\mathrm{c}}$ | $4\pi k_{\mathrm{cs}} r_{\mathrm{cs}}$ | $4\pi k_{\mathrm{ct}} r_{\mathrm{ct}}$ |
|---|---|---|---|---|---|---|
| 8 | 0.8143 | 0.3603 | 0.4539 | 0.0578 | 0.0567 | 0.0587 |
| 12 | 0.5143 | 0.2456 | 0.2686 | 0.0350 | 0.0369 | 0.0335 |
| 16 | 0.3784 | 0.1854 | 0.1930 | 0.0254 | 0.0274 | 0.0238 |
| 24 | 0.2488 | 0.1241 | 0.1247 | 0.0166 | 0.0182 | 0.0152 |
| 32 | 0.1857 | 0.0932 | 0.0925 | 0.0123 | 0.0136 | 0.0113 |
| 48 | 0.1234 | 0.0622 | 0.0612 | 0.0082 | 0.0090 | 0.0074 |

Table 3. Values of $r'_{\mathrm{c}}, r'_{\mathrm{cs}}, r'_{\mathrm{ct}}$ (Wilson action, open-SF bc)

| $L$ | $r'_{\mathrm{c}} \times 10^4$ | $r'_{\mathrm{cs}} \times 10^4$ | $r'_{\mathrm{ct}} \times 10^4$ | $4\pi k_{\mathrm{c}} r'_{\mathrm{c}}$ | $4\pi k_{\mathrm{cs}} r'_{\mathrm{cs}}$ | $4\pi k_{\mathrm{ct}} r'_{\mathrm{ct}}$ |
|---|---|---|---|---|---|---|
| 8 | 1.5146 | 0.7632 | 0.7514 | 0.1075 | 0.1201 | 0.0971 |
| 12 | 0.9660 | 0.4996 | 0.4664 | 0.0658 | 0.0750 | 0.0581 |
| 16 | 0.7124 | 0.3708 | 0.3416 | 0.0479 | 0.0549 | 0.0421 |
| 24 | 0.4690 | 0.2451 | 0.2239 | 0.0312 | 0.0359 | 0.0274 |
| 32 | 0.3502 | 0.1832 | 0.1670 | 0.0233 | 0.0267 | 0.0203 |
| 48 | 0.2327 | 0.1218 | 0.1108 | 0.0154 | 0.0177 | 0.0135 |

Table 4. Values of $r_{\rm p}, r_{\rm ps}, r_{\rm pt}$ (Wilson action, SF bc)

| $L$ | $r_{\rm p} \times 10^4$ | $r_{\rm ps} \times 10^4$ | $r_{\rm pt} \times 10^4$ | $4\pi k_{\rm p} r_{\rm p}$ | $4\pi k_{\rm ps} r_{\rm ps}$ | $4\pi k_{\rm pt} r_{\rm pt}$ |
|---|---|---|---|---|---|---|
| 8 | 6.6960 | 1.8498 | 4.8462 | 0.3816 | 0.2128 | 0.5473 |
| 12 | 4.3489 | 1.0787 | 3.2701 | 0.2884 | 0.1445 | 0.4295 |
| 16 | 3.2394 | 0.7717 | 2.4676 | 0.2240 | 0.1078 | 0.3379 |
| 24 | 2.1496 | 0.4972 | 1.6524 | 0.1528 | 0.0714 | 0.2326 |
| 32 | 1.6096 | 0.3684 | 1.2412 | 0.1155 | 0.0534 | 0.1764 |
| 48 | 1.0718 | 0.2435 | 0.8284 | 0.0774 | 0.0355 | 0.1185 |

Table 5. Values of $r_{\rm p}, r_{\rm ps}, r_{\rm pt}$ (Wilson action, open-SF bc)

| $L$ | $r_{\rm p} \times 10^4$ | $r_{\rm ps} \times 10^4$ | $r_{\rm pt} \times 10^4$ | $4\pi k_{\rm p} r_{\rm p}$ | $4\pi k_{\rm ps} r_{\rm ps}$ | $4\pi k_{\rm pt} r_{\rm pt}$ |
|---|---|---|---|---|---|---|
| 8 | 0.9717 | 0.4405 | 0.5312 | 0.0512 | 0.0501 | 0.0522 |
| 12 | 0.5577 | 0.2671 | 0.2906 | 0.0339 | 0.0354 | 0.0326 |
| 16 | 0.3964 | 0.1941 | 0.2022 | 0.0250 | 0.0268 | 0.0235 |
| 24 | 0.2541 | 0.1266 | 0.1275 | 0.0165 | 0.0180 | 0.0152 |
| 32 | 0.1879 | 0.0942 | 0.0937 | 0.0123 | 0.0135 | 0.0113 |
| 48 | 0.1240 | 0.0625 | 0.0615 | 0.0082 | 0.0090 | 0.0074 |

Table 6. Values of $r'_{\rm p}, r'_{\rm ps}, r'_{\rm pt}$ (Wilson action, open-SF bc)

| $L$ | $r'_{\rm p} \times 10^4$ | $r'_{\rm ps} \times 10^4$ | $r'_{\rm pt} \times 10^4$ | $4\pi k_{\rm p} r'_{\rm p}$ | $4\pi k_{\rm ps} r'_{\rm ps}$ | $4\pi k_{\rm pt} r'_{\rm pt}$ |
|---|---|---|---|---|---|---|
| 8 | 1.8005 | 0.9320 | 0.8685 | 0.0949 | 0.1061 | 0.0853 |
| 12 | 1.0415 | 0.5429 | 0.4986 | 0.0633 | 0.0720 | 0.0559 |
| 16 | 0.7435 | 0.3883 | 0.3553 | 0.0470 | 0.0537 | 0.0413 |
| 24 | 0.4781 | 0.2501 | 0.2280 | 0.0310 | 0.0356 | 0.0272 |
| 32 | 0.3540 | 0.1853 | 0.1687 | 0.0232 | 0.0266 | 0.0203 |
| 48 | 0.2338 | 0.1224 | 0.1114 | 0.0154 | 0.0177 | 0.0135 |