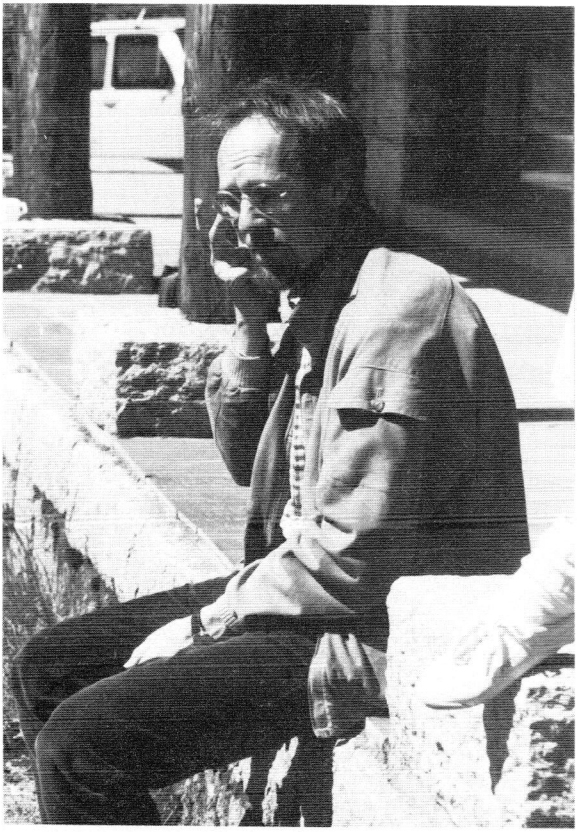
COURSE 8

SELECTED TOPICS IN LATTICE FIELD THEORY

M. LÜSCHER

Deutsches Elektronen-Synchrotron DESY,
D-2000 Hamburg 52, FRG

E. Brézin and J. Zinn-Justin, eds.
Les Houches, Session XLIX, 1988
Champs, Cordes et Phénomènes Critiques
/Fields, Strings and Critical Phenomena



Contents

1. Introduction

Over the past few years a lot of work has been invested to develop lattice gauge theories, the ultimate goal being to resolve a number of puzzles in elementary particle physics that cannot be addressed in a conventional perturbative framework [1]. In particular, one hopes to calculate the hadron spectrum and other low energy properties of quantum chromodynamics (QCD) through a numerical simulation of lattice QCD. This is a very ambitious program, which requires a lot of "technical" insight and enormous computer resources. Progress is slow therefore and one should not expect reliable results to come out soon. Other interesting field theories such as scalar theories, Higgs models and pure non-Abelian gauge theories, are however more amenable to numerical simulations and this has led (and still leads) to a greatly improved qualitative and quantitative understanding of some of these systems.

The enthusiasm of lattice gauge theorists is mainly based on the fact that numerical simulations provide a novel and widely applicable tool to solve field theories that are otherwise untractable. But there are also many other good reasons to study lattice field theories. An important observation is, for example, that apart perhaps from stochastic quantization [2,3], lattice gauge theories are actually the only known consistent regularization of non-Abelian gauge theories. Of course, in perturbation theory one may alternatively use dimensional regularization [4,5] or the BPHZL [26-29] finite part prescription, but these methods are strictly perturbative and do not define the theory before one expands in powers of the coupling constant.

Since a lattice theory is mathematically well-defined, it is also a good starting point to derive properties of the model in a rigorous way or even to "construct" the corresponding continuum theory [6,7]. Moreover, conceptual questions can often be addressed on a more concrete level on the lattice, than would be possible in a formal continuum approach (quark confinement, the Higgs mechanism and the issue of "triviality" [8], for example).

In these lectures I would like to discuss three fundamental questions, listed below, which arise in any lattice quantum field theory and which should be answered mainly to gain conceptual clarity (some of the results presented are, however, also of immediate practical value).

(1) In the most common formulation of lattice quantum field theory, the lattice is a euclidean space-time lattice and the fields are (classical) random variables associated with the elements of the lattice. The quantum mechanical interpretation of the theory then relies on the well-known transfer matrix construction [9-17]. In particular, the notion of a stable particle has an unambiguous meaning in this framework for any value of the lattice spacing. What is less clear in this basically euclidean setting, is how the scattering matrix for these particles can be defined in a physically sensible way. Of course, one could always first take the continuum limit and then apply one of the standard procedures (LSZ [18] or Haag-Ruelle [19-22]). But this is often impractical, in particular, in numerical simulations the lattice spacing is never really small compared to the physical scales in the model studied. Thus, a lattice scattering theory is desirable and, using some recent results of Barata and Fredenhagen [23,24], I shall show in section 2 how this can be done in a nice way.

(2) It is usually not difficult to derive the weak coupling perturbation expansion in lattice quantum field theories, but the Feynman rules which one obtains are in all cases rather complicated, especially, the propagators are not rational functions of momentum. Thus, the lattice regularisation is unpractical for explicit high-order perturbative calculations. Perturbation theory is, however, very useful to study the continuum limit of lattice theories, i.e. to see whether the renormalization procedure works in the expected way, whether Lorentz invariance gets restored in the continuum limit and whether anomalies are properly reproduced, to mention just a few examples. The appropriate tool to tackle these questions is the lattice power counting theorem which has recently been established by Reisz [34–36]. In section 3, I review this result and I also explain, why the BRS-symmetry [39,40] (which is instrumental to the proof of renormalizability of non-Abelian gauge theories) is unaffected by the lattice [41,42].

(3) Due to the limitations of present day computers, numerical simulations of lattice field theories are restricted to rather small systems with a few 100,000 lattice points at most. Thus, the space-time volumes which can be accommodated are often not very much larger than the physical scales in the theory and the results of the computations must

therefore be expected to depend on the size of the lattice. Such finite size effects have been observed in several numerical experiments. On the theoretical side, the situation is not fully understood. Especially in (pure) non-Abelian gauge theories the volume dependence is very complicated with several regimes depending on the ratio of the lattice size and the dynamically generated correlation length. Precise formulae exist, on the other hand, describing the final asymptotic approach of particle masses and other quantities of interest to the infinite volume limit [44-57]. In the third part of my lectures (section 4), the aim is mainly to illustrate the above remarks and to provide an intuitive understanding for the dynamical origins of the various finite volume effects.

Acknowledgements

While preparing these lectures, I have benefitted from discussions with many colleagues and students. In particular, I am indebted to Klaus Fredenhagen for explaining me some of the technical details of the lattice scattering theory, Istvan Montvay and Gernot Münster for discussions on finite size effects in models with spontaneously broken symmetries, Pierre van Baal for preparing a table of glueball masses and Peter Weisz for a critical reading of the manuscript. I would also like to thank Edouard Brézin and Jean Zinn-Justin for their kind invitation and the perfect organization of the School.

2. Particle scattering in euclidean lattice theories

2.1. Construction and interpretation of the transfer matrix

The transfer matrix formalism [9-17] is of course well-known – it now even appears in text books – but I need to go through this old material to prepare the ground for the discussion of particle scattering. For concreteness, I shall develop the concepts for a definite model, namely pure SU(N) gauge theory on a 4-dimensional hypercubic lattice. The methods are however more generally applicable and carry over to essentially any theory with only massive particles.

An SU(N) lattice gauge field is an assignment of a matrix $U(x,\mu) \in \mathrm{SU}(N)$ to every lattice bond with endpoints x and $x+\hat{\mu}$, where $\hat{\mu}$ denotes the unit vector in the positive μ-direction (the lattice spacing is set equal to one, i.e. $x \in \mathrm{Z}^4$, and $\mu = 0,1,2,3$). Initially, I shall assume that the euclidean space-time lattice Λ is finite with a rectangular shape $T \times L \times L \times L$, where T refers to time and L to space. The boundary

conditions on the gauge fields are taken periodic, viz.

$$\begin{aligned} U(x+T\hat{\nu},\mu) &= U(x,\mu), \qquad (\nu=0), \\ U(x+L\hat{\nu},\mu) &= U(x,\mu), \qquad (\nu>0). \end{aligned} \tag{2.1}$$

The dynamics of the model is specified by the Wilson action, which may be written in the form

$$S[U] = \frac{1}{g_0^2} \sum_{x\in\Lambda} \sum_{\mu,\nu} \mathcal{P}_{\mu\nu}(x), \tag{2.2}$$

where $\mathcal{P}_{\mu\nu}$ denotes the plaquette field,

$$\mathcal{P}_{\mu\nu}(x) = \mathrm{Re}\,\mathrm{Tr}\{1 - U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1}\}, \tag{2.3}$$

and g_0 is the bare coupling constant. The field $\mathcal{P}_{\mu\nu}(x)$ and hence the action $S[U]$ are invariant under the gauge transformation

$$U(x,\mu) \to U^g(x,\mu) = g(x)U(x,\mu)g(x+\hat{\mu})^{-1} \tag{2.4}$$

for any field $g(x) \in \mathrm{SU}(N)$ which respects the periodicity of the lattice Λ.

The objects from which all the physics described by the model is to be extracted ultimately, are the expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{x\in\Lambda} \prod_{\mu=0}^{3} \mathrm{d}U(x,\mu) \mathcal{O}[U] \mathrm{e}^{-S[U]}, \tag{2.5}$$

$$Z = \int \prod_{x\in\Lambda} \prod_{\mu=0}^{3} \mathrm{d}U(x,\mu) \mathrm{e}^{-S[U]}, \tag{2.6}$$

where $\mathcal{O}[U]$ (the "observable") is any gauge invariant combination of the field variables $U(x,\mu)$ and $\mathrm{d}U(x,\mu)$ denotes the invariant measure on $\mathrm{SU}(N)$. In the above formulation, lattice gauge theories look like a classical statistical mechanical system with partition function Z and thermal averages $\langle \mathcal{O} \rangle$. The purpose of the transfer matrix construction is to provide an equivalent representation of the model as a quantum mechanical system in 3 dimensions with a Hilbert space $\mathcal{H}$ of physical states, a Hamilton operator H and linear operators $\hat{\mathcal{O}}$ corresponding to the euclidean "observables" $\mathcal{O}$.

The definition of this quantum mechanical system is as follows. Let Λ_0 be the $L \times L \times L$ spacial sublattice of Λ at time $x_0 = 0$. We then consider wave functions $\psi[V]$, where the argument $V(\mathbf{x}, k)$ ($\mathbf{x} \in \Lambda_0$, $k = 1, 2, 3$) runs through all SU(N) gauge fields on Λ_0. A natural scalar product for such wave functions is

$$(\phi, \psi) = \int \prod_{\mathbf{x} \in \Lambda_0} \prod_{k=1}^{3} \mathrm{d}V(\mathbf{x}, k)\, \phi[V]^* \psi[V]. \tag{2.7}$$

The physical Hilbert space $\mathcal{H}$ is the space of all normalizable wave functions $\psi[V]$, which are gauge invariant, i.e. which satisfy

$$\psi[V^g] = \psi[V] \tag{2.8}$$

for all gauge transformations $g(\mathbf{x})$, $\mathbf{x} \in \Lambda_0$.

We now proceed to define the transfer matrix T from which the Hamilton operator H will later be derived. T is an integral operator acting in $\mathcal{H}$, viz.

$$(\mathrm{T}\psi)[V] = \int \prod_{\mathbf{x} \in \Lambda_0} \prod_{k=1}^{3} \mathrm{d}V'(\mathbf{x}, k)\, K[V, V']\psi[V'], \tag{2.9}$$

where the integral kernel $K[V, V']$ is given by

$$K[V, V'] = \int \prod_{\mathbf{x} \in \Lambda_0} \mathrm{d}W(\mathbf{x}) \exp -\Delta S[V, W, V'], \tag{2.10}$$

$$\Delta S = \frac{1}{g_0^2} \sum_{\mathbf{x} \in \Lambda_0} \Big\{ 2 \sum_k \mathcal{Q}_{0k}(\mathbf{x}) + \sum_{k<l} (\mathcal{Q}_{kl}(\mathbf{x}) + \mathcal{Q}'_{kl}(\mathbf{x})) \Big\}. \tag{2.11}$$

Here, $\mathcal{Q}_{kl}$ and $\mathcal{Q}'_{kl}$ are defined in the same way as $\mathcal{P}_{kl}$ (eq. (2.3)) with U replaced by V and V', respectively, while $\mathcal{Q}_{0k}$ is given by

$$\mathcal{Q}_{0k}(\mathbf{x}) = \mathrm{ReTr}\{1 - V'(\mathbf{x}, k) W(\mathbf{x} + \hat{k}) V(\mathbf{x}, k)^{-1} W(\mathbf{x})^{-1}\}. \tag{2.12}$$

Thus, ΔS is equal to the action on the double layer of equal time hyperplanes shown in fig. 1, except that the space-like plaquettes are given the weight $\frac{1}{2}$.

From the above it should be rather obvious that the fundamental identity

$$Z = \mathrm{Tr}\{\mathrm{T}^T\} \tag{2.13}$$

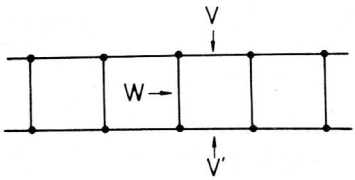


Fig. 1. Side view of a double layer of lattices Λ_0. The gauge field $V(\mathbf{x}, k)$ lives on the upper layer and $V'(\mathbf{x}, k)$ on the lower one, while the auxiliary field $W(\mathbf{x})$ integrated over in eq. (2.10) sits on the time-like links connecting the two layers.

holds, where "Tr" means the trace in the Hilbert space $\mathcal{H}$. Moreover, the euclidean correlation functions $\langle A(x)B(y)...\rangle$ of local, gauge invariant fields $A(x)$, $B(y)$,... can also be given a quantum mechanical interpretation. The situation is especially simple, when these fields are polynomials of the gauge field variables $U(x, k)$ at a fixed time x_0 (such as the plaquette field $\mathcal{P}_{kl}(x)$, for example). If $A(x)$ is such a field, we may define an associated operator $\hat{A}(\mathbf{x})$ through

$$(\hat{A}(\mathbf{x})\psi)[V] = A(0, \mathbf{x})\psi[V], \tag{2.14}$$

where on the r.h.s. $A(0, \mathbf{x})$ is to be evaluated for the gauge field $V(\mathbf{x}, k)$. For $0 \leq x_0 \leq T$ we then have

$$\langle A(x)A(0)\rangle = \frac{1}{Z}\mathrm{Tr}\{\mathrm{T}^{T-x_0}\hat{A}(\mathbf{x})\mathrm{T}^{x_0}\hat{A}(\mathbf{0})\} \tag{2.15}$$

and a similar formula holds for the higher correlation functions and the correlation functions involving different fields.

For fields $A(x)$ which are combinations of gauge field variables from several time slices, a representation like eq. (2.15) can still be derived, but the definition of the associated operator $\hat{A}(\mathbf{x})$ is more complicated in this case. For example, the plaquette operator $\hat{\mathcal{P}}_{0k}$ is an integral operator with a kernel given by eqs. (2.10), (2.11), where the integrand in eq. (2.10) is to be multiplied by $\mathcal{Q}_{0k}(\mathbf{x})$. Equation (2.15) then holds again if we reduce the powers of T by one to account for the fact that $\hat{\mathcal{P}}_{0k}$ already includes one time step.

The transfer matrix T is a bounded, self-adjoint operator which has a completely discrete spectrum (for finite L). Moreover, all its eigenvalues are strictly positive and the maximal eigenvalue λ_0 is not degenerate [11,13-15]. Thus, the Hamilton operator

$$\mathrm{H} = -\ln(\mathrm{T}/\lambda_0) \tag{2.16}$$

is a well-defined (unbounded) self-adjoint operator in $\mathcal{H}$. As usual, H is interpreted as the energy operator of the model. By construction, $\mathrm{H} \geq 0$ and there is a unique state $|0\rangle$, the "vacuum", with $\mathrm{H}|0\rangle = 0$.

From the discussion so far it follows that the limit $T \to \infty$ of the expectation values $\langle \mathcal{O} \rangle$ exists when $\mathcal{O}$ is a polynomial of the field variables $U(x, \mu)$. In particular, for the 2-point function (2.15) we have, at $T = \infty$,

$$\langle A(x)A(0) \rangle = \langle 0|\hat{A}(\mathbf{x})\mathrm{e}^{-x_0\mathrm{H}}\hat{A}(\mathbf{0})|0\rangle. \tag{2.17}$$

The limit $L \to \infty$ is less trivial and I shall not discuss it here, but just assume it exists (as is the case for sufficiently strong coupling g_0 [13,14]).

2.2. Localized one-particle states

For any positive value of the bare coupling g_0, one expects that in (pure) SU(N) lattice gauge theories the infinite volume ground state is unique and that the model describes a number of interacting, massive stable particles, the glueballs. The evidence for this expectation is not overwhelming, but it seems to be the logically simplest possibility given the strong coupling results and the asymptotic freedom of the theory. Anyway, I shall here assume that this is the situation for the value of g_0 considered and develop the scattering theory for this case.

According to the most recent numerical simulations of the model (cf. subsection 4.4), the lightest glueball is a singlet under lattice rotations and the corresponding one-particle states $|\mathbf{p}\rangle$ can therefore be labelled by a momentum $\mathbf{p}$ in the first Brillouin zone. These (improper) states are simultaneous eigenstates of H and of the unitary operators $\mathrm{U}(\mathbf{a})$ which represent the translations by integer vectors $\mathbf{a}$:

$$\begin{aligned} \mathrm{H}|\mathbf{p}\rangle &= \omega(\mathbf{p})|\mathbf{p}\rangle, \qquad |p_k| \leq \pi, \\ \mathrm{U}(\mathbf{a})|\mathbf{p}\rangle &= \mathrm{e}^{-i\mathbf{p}\mathbf{a}}|\mathbf{p}\rangle. \end{aligned} \tag{2.18}$$

Here, the one-particle energy $\omega(\mathbf{p})$ is some positive periodic function and

$$m = \omega(\mathbf{0}) \tag{2.19}$$

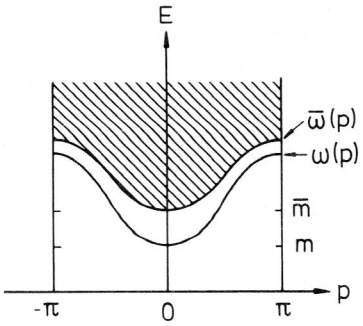


Fig. 2. Qualitative plot of the energy-momentum spectrum showing the isolated energy shell $E = \omega(\mathbf{p})$ of the lightest glueball. The energy shells of the other stable particles and the multi-particle continuum are contained in the shaded area.

is interpreted as the mass of the particle. The energy-momentum spectrum at low energies is thus expected to look qualitatively as in fig. 2.

The normalization of the one-particle states may be chosen such that

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2\omega(\mathbf{p})(2\pi)^3\delta_P(\mathbf{p}-\mathbf{q}), \tag{2.20}$$

where δ_P denotes the periodic δ-function. As an "interpolating" field, one may take the plaquette field

$$\mathcal{P}(x) = \sum_{k<l}\{\mathcal{P}_{kl}(x) - \langle \mathcal{P}_{kl}(x)\rangle\}, \tag{2.21}$$

and, by adjusting the phase of the one-particle states, one may assume that the associated field renormalization constant $Z(\mathbf{p})$, defined through

$$\langle 0|\hat{\mathcal{P}}(\mathbf{0})|\mathbf{p}\rangle = \sqrt{Z(\mathbf{p})}, \tag{2.22}$$

is real and positive (I have taken it for granted that the matrix element (2.22) does not vanish). With these notations, the Källen-Lehmann

representation of the two-point function of the plaquette field $\mathcal{P}(x)$ reads (cf. eq. (2.17))

$$\begin{aligned} \langle \mathcal{P}(x)\mathcal{P}(0)\rangle &= \int_m^\infty \mathrm{d}E \int_{-\pi}^{\pi} \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{-|x_0|E+\mathrm{i}\mathbf{p}\mathbf{x}}\rho(E,\mathbf{p}), \\ \rho(E,\mathbf{p}) &= \frac{Z(\mathbf{p})}{2\omega(\mathbf{p})}\delta(E-\omega(\mathbf{p})) + \bar{\rho}(E,\mathbf{p}), \end{aligned} \tag{2.23}$$

where $\bar{\rho}(E,\mathbf{p}) \geq 0$ is supported in the shaded area of fig. 2.

Normalizable one-particle states (wave packets) can be formed by

$$|f\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\sqrt{Z(\mathbf{p})}}{2\omega(\mathbf{p})} f(\mathbf{p})|\mathbf{p}\rangle, \tag{2.24}$$

where $f(\mathbf{p})$ is any smooth periodic function and the factor $\sqrt{Z}$ is included for later convenience. If $f(\mathbf{p})$ is such that $\langle f|f\rangle = 1$, the interpretation of eq. (2.24) is that $|f\rangle$ describes a one-particle state with a probability distribution of momentum equal to

$$\frac{\mathrm{d}^3p}{(2\pi)^3} \frac{Z(\mathbf{p})}{2\omega(\mathbf{p})} |f(\mathbf{p})|^2. \tag{2.25}$$

An important and rather non-trivial observation now is that a glueball in the state $|f\rangle$ is (essentially) localized in a bounded region $\mathcal{R}$ of space.

The precise meaning of this statement is the following. Suppose $\hat{A}(\mathbf{x})$ is a local field such as the plaquette field $\hat{\mathcal{P}}_{kl}(\mathbf{x})$ or any other "observable" and assume $\mathbf{x}$ is far away from the region $\mathcal{R}$. A measurement of this quantity in the state $|f\rangle$ gives the value $\langle f|\hat{A}(\mathbf{x})|f\rangle$ in the average. If the particle described by $|f\rangle$ is essentially confined to the region $\mathcal{R}$, one expects that the measurement of $\hat{A}(\mathbf{x})$ should give the same result as in the vacuum state up to a small deviation. Thus, the glueball in the state $|f\rangle$ is well-localized, if for all local polynomial fields $A(x)$ one has

$$\langle f|\hat{A}(\mathbf{x})|f\rangle = \langle 0|\hat{A}(\mathbf{x})|0\rangle + \varepsilon(\mathbf{x}), \tag{2.26}$$

where $\varepsilon(\mathbf{x})$ is rapidly vanishing for $|\mathbf{x}| \to \infty$.

The proof that the state $|f\rangle$ as defined by eq. (2.24) does indeed have this property is a bit technical, but I shall present it here in some detail, because it involves an argument [23,24] which is really the key to the scattering theory developed later. For simplicity, I shall only discuss

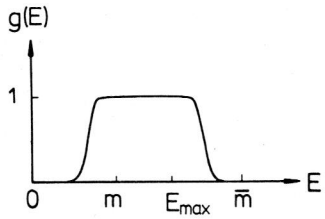


Fig. 3. Plot of the cutoff function $g(E)$.

the case where the glueball is at rest, i.e. where $f(\mathbf{p})$ vanishes outside a small neighborhood of $\mathbf{p} = \mathbf{0}$. The maximal value $E_{\max}$ of the one-particle energy $\omega(\mathbf{p})$ for $\mathbf{p}$ in the support of f will then be below the energy $\overline{m}$ where the shaded area in fig. 2 begins. Now we choose a smooth cutoff function $g(E) \geq 0$ with a shape as in fig. 3 and it is then possible to show that

$$|f\rangle = \sum_{\mathbf{y}} \tilde{f}(\mathbf{y}) g(\mathrm{H}) \hat{\mathcal{P}}(\mathbf{y}) |0\rangle, \tag{2.27}$$

where $\tilde{f}$ is the Fourier transform of f. Indeed, because the multi-particle states have an energy of at least $\overline{m}$, the operator $g(\mathrm{H})$ projects any state on a one-particle state. To prove eq. (2.27), it is therefore sufficient (and trivial) to work out the scalar product of the r.h.s. with the one-particle basis vector $|\mathbf{q}\rangle$ and to verify that it is equal to $\langle \mathbf{q}|f\rangle$ for all $\mathbf{q}$.

Our task is to show that the deviation $\varepsilon(\mathbf{x})$ defined through eq. (2.26) is rapidly decaying for $|\mathbf{x}| \to \infty$. The strategy in what follows basically is to approximate the matrix element $\langle f|\hat{A}(\mathbf{x})|f\rangle$ by euclidean correlation functions and then to use the exponential clustering properties of these functions as they derive from spectral representations of the type (2.23) (with space and time interchanged).

Combining eqs. (2.26), (2.27) and $\langle f|f\rangle = 1$, we have

$$\begin{aligned}\varepsilon(\mathbf{x}) = \sum_{\mathbf{y},\mathbf{z}} \tilde{f}(\mathbf{y})^* \tilde{f}(\mathbf{z}) \big\{ &\langle 0|\hat{\mathcal{P}}(\mathbf{y}) g(\mathrm{H}) \hat{A}(\mathbf{x}) g(\mathrm{H}) \hat{\mathcal{P}}(\mathbf{z})|0\rangle \\ &- \langle 0|\hat{\mathcal{P}}(\mathbf{y}) g(\mathrm{H})^2 \hat{\mathcal{P}}(\mathbf{z})|0\rangle \langle 0|\hat{A}(\mathbf{x})|0\rangle \big\}. \end{aligned} \tag{2.28}$$

For a local polynomial field $A(x)$, the associated operator $\hat{A}(\mathbf{x})$ is bounded and so are the other operators in eq. (2.28). Furthermore, $\tilde{f}(\mathbf{y})$ is rapidly vanishing for $|\mathbf{y}| \to \infty$ and the summations over $\mathbf{y}$ and $\mathbf{z}$ can therefore be restricted to the cube

$$\mathcal{C}(\mathbf{x}) = \{\mathbf{y} \in \mathrm{Z}^3 \mid |y_k| \leq r(\mathbf{x})\}, \tag{2.29}$$

$$r(\mathbf{x}) = \frac{1}{2} \max_k |x_k|, \tag{2.30}$$

if we allow for an error of order $|\mathbf{x}|^{-s}$, where s is any integer.

We next consider the operator $g(\mathrm{H})$ which is a well-defined but rather abstract object. The idea here is to approximate it by polynomials in the transfer matrix [23,24]. To this end first note that $g(E)$ is a smooth function of $\lambda = \mathrm{e}^{-E}$ in the interval $0 \leq \lambda \leq 1$. It can therefore be expanded in a uniformly and rapidly convergent series of Chebyshev polynomials in λ (for a very readable intoduction to the basic properties and uses of Chebyshev polynomials see [25]). When truncated at order K, a polynomial approximation

$$g(E) = \sum_{k=0}^{K} a_k^{(K)} \mathrm{e}^{-kE} + \delta_K(E) \tag{2.31}$$

is obtained, where the error δ_K is bounded by

$$|\delta_K(E)| \leq C_s / K^s \tag{2.32}$$

for any power s. Furthermore, there exist constants a, b such that for all $K \geq 1$, the coefficients $a_k^{(K)}$ satisfy

$$\sum_{k=0}^{K} |a_k^{(K)}| \leq a \mathrm{e}^{bK}. \tag{2.33}$$

These bounds (eqs. (2.32),(2.33)) are just general properties of the Chebyshev expansion [25], which follow from the assumed smoothness of g and the definition of the Chebyshev polynomials. The constants C_s and a depend on the detailed properties of g while b is related to the growth of the coefficients of the Chebyshev polynomials as a function of their order and can hence be chosen independently of g.

If one now inserts eq. (2.31) in eq. (2.28), one obtains a 3-point function of the operators $\hat{A}(\mathbf{x})$ and $\hat{\mathcal{P}}(\mathbf{y})$ which can be rewritten in the form of a euclidean correlation function of the associated fields $A(x)$ and $\mathcal{P}(y)$. The result then is

$$\varepsilon(\mathbf{x}) = \sum_{k,j=0}^{K} a_k^{(K)} a_j^{(K)} \sum_{\mathbf{y},\mathbf{z}\in\mathcal{C}(\mathbf{x})} \tilde{f}(\mathbf{y})^* \tilde{f}(\mathbf{z}) \langle A(x)\mathcal{P}(y)\mathcal{P}(z)\rangle^{\mathrm{con}} + O(K^{-s}) + O(|\mathbf{x}|^{-s}), \tag{2.34}$$

where $x_0 = 0$, $y_0 = k$, $z_0 = -j$ and the connected part of the 3-point function is defined by

$$\langle A(x)\mathcal{P}(y)\mathcal{P}(z)\rangle^{\mathrm{con}} = \langle A(x)\mathcal{P}(y)\mathcal{P}(z)\rangle - \langle A(x)\rangle\langle\mathcal{P}(y)\mathcal{P}(z)\rangle \tag{2.35}$$

(recall that $\langle\mathcal{P}(y)\rangle = 0$).

In the final step of the argument we make use of the (discrete) euclidean rotation invariance of the theory to prove that

$$|\langle A(x)\mathcal{P}(y)\mathcal{P}(z)\rangle^{\mathrm{con}}| \leq C\exp(-mr(\mathbf{x})) \tag{2.36}$$

for some constant C and all x, y, z with $\mathbf{y}, \mathbf{z} \in \mathcal{C}(\mathbf{x})$ (cf. eqs. (2.29), (2.30)). Indeed, there are always at least $r(\mathbf{x})$ lattice planes between $\mathbf{x}$ and $\mathcal{C}(\mathbf{x})$ and if we declare the direction orthogonal to these planes to be the "time" direction, the bound (2.36) follows immediately from the quantum mechanical representation, analogous to eq. (2.17), of the 3-point function and the trivial estimate

$$\| \mathrm{e}^{-r\mathrm{H}} - |0\rangle\langle 0| \| \leq \mathrm{e}^{-mr}. \tag{2.37}$$

Finally, we combine eqs. (2.33)-(2.36) and obtain the bound

$$|\varepsilon(\mathbf{x})| \leq C' \exp\{2bK - mr(\mathbf{x})\} + O(K^{-s}) + O(|\mathbf{x}|^{-s}), \tag{2.38}$$

which becomes $|\varepsilon(\mathbf{x})| \leq O(|\mathbf{x}|^{-s})$ if we let K grow proportional to $r(\mathbf{x})$ with a rate smaller than $m/2b$. To sum up, we have shown that $\varepsilon(\mathbf{x})$ vanishes faster than any inverse power of $|\mathbf{x}|$ and the glueball in the state $|f\rangle$ is therefore well localized.

2.3. Scattering states

The basic problem in scattering theory is to construct states which, for large times, describe a number of localized particles (wave packets) running away from each other. For simplicity, I shall only discuss the case of elastic scattering of the lightest glueballs, although the generalization to other channels is straightforward. As a preparation for the construction of the corresponding in- and out-going scattering states it is useful to discuss how the physical situation of two well-localized, widely separated glueballs, characterized by wave functions $f_1(\mathbf{p})$ and $f_2(\mathbf{p})$, can be approximately described by a state $|f_1, f_2\rangle$ in the Hilbert space $\mathcal{H}$.

By an argumentation similar to the one applied in case of localized one-particle states, the basic property of $|f_1, f_2\rangle$ is expected to be that for all local polynomial fields $A(x)$, $B(y)$ one has

$$\langle f_1, f_2|\hat{A}(\mathbf{x})\hat{B}(\mathbf{y})|f_1, f_2\rangle = \langle f_1|\hat{A}(\mathbf{x})|f_1\rangle\langle f_2|\hat{B}(\mathbf{y})|f_2\rangle + \cdots + \varepsilon(d) \tag{2.39}$$

with an error $\varepsilon(d)$ which vanishes rapidly as the distance d between the wave packets f_1 and f_2 is made large (the dots in eq. (2.39) indicate a number of terms as they arise from the cluster decomposition of the matrix element on the l.h.s.). States satisfying eq. (2.39) can be constructed by

$$|f_1, f_2\rangle = a^\dagger(f_1)a^\dagger(f_2)|0\rangle, \tag{2.40}$$

where the Haag-Ruelle creation operator $a^\dagger(f)$ is defined as follows [23,24]. Let $g(E, \mathbf{p}) \geq 0$ be a smooth cutoff function which is equal to 1 in a neighborhood of the one-particle mass shell $E = \omega(\mathbf{p})$ and which vanishes for $E \leq 0$ and $E \geq \bar{\omega}(\mathbf{p})$, where $\bar{\omega}(\mathbf{p})$ is the (lower) boundary of the shaded area in fig. 2. For $t \in \mathrm{R}$, $\mathbf{x} \in \mathrm{Z}^3$ and any test-function $f(\mathbf{p})$ define

$$\hat{f}(t, \mathbf{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \mathrm{e}^{i(\mathbf{px}-Et)} g(E, \mathbf{p}) f(\mathbf{p}). \tag{2.41}$$

The definition of $a^\dagger(f)$ then reads

$$a^\dagger(f) = \int \mathrm{d}t \sum_{\mathbf{x}} \hat{f}(t, \mathbf{x}) \, \mathrm{e}^{\mathrm{i}Ht} \hat{\mathcal{P}}(\mathbf{x}) \, \mathrm{e}^{-\mathrm{i}Ht}. \tag{2.42}$$

It is easy to verify that $a^\dagger(f)$ creates a one-particle state from the vacuum, viz.

$$a^\dagger(f)|0\rangle = |f\rangle. \tag{2.43}$$

The adjoint operator $a(f)$, on the other hand, satisfies

$$\begin{aligned} a(f)|0\rangle &= 0, \\ a(f_1)a^\dagger(f_2)|0\rangle &= |0\rangle\langle f_1|f_2\rangle, \end{aligned} \tag{2.44}$$

and is therefore a one-particle annihilation operator. In view of these properties, the Ansatz (2.40) for a two-particle state looks promising, but one still has to show that (2.39) is indeed fulfilled when the wave packets f_1 and f_2 are shifted away from each other. The techniques that we have applied to prove the localization of one-particle states can however easily be generalized to the present case. The only new aspect is perhaps that the matrix element (2.39) involves the time evolution operator $\mathrm{e}^{-\mathrm{i}Ht}$ at several places, but it always appears in conjunction with a cutoff function g and what one does then is to approximate the product of the two (rather than g alone) by polynomials in the transfer matrix.

I would now like to proceed to discuss in- and out-going two-particle scattering states. The physical characterization of an out-going state $|f_1, f_2 \text{ out}\rangle$ is that for large times t, it evolves into a state describing two widely separated glueballs which move away from each other. The motion of a single glueball is determined by the energy-momentum relation $E = \omega(\mathbf{p})$, i.e.

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}Ht}|f\rangle &= |f_t\rangle, \\ f_t(\mathbf{p}) &= \mathrm{e}^{-\mathrm{i}\omega(\mathbf{p})t} f(\mathbf{p}). \end{aligned} \tag{2.45}$$

From the discussion above of localized two-particle states, it thus follows that an accurate description of the envisaged out-going situation is given by the state

$$a^\dagger(f_{1,t})a^\dagger(f_{2,t})|0\rangle, \tag{2.46}$$

provided only that the wave packets $|f_{1,t}\rangle$ and $|f_{2,t}\rangle$ are indeed running away from each other. Using stationary phase methods, it is possible to show that this is the case if $f_1(\mathbf{p})$ and $f_2(\mathbf{p})$ have compact support and if the corresponding ranges of group velocities $\nabla\omega(\mathbf{p})$ do not overlap.

Summarizing the discussion so far, one expects that for wave packets f_1, f_2 with the above properties the asymptotic relation

$$\mathrm{e}^{-\mathrm{i}Ht}|f_1, f_2 \text{ out}\rangle \underset{t\to\infty}{\sim} a^\dagger(f_{1,t})a^\dagger(f_{2,t})|0\rangle. \tag{2.47}$$

holds and hence

$$|f_1, f_2 \text{ out}\rangle = \lim_{t\to\infty} e^{iHt} a^\dagger(f_{1,t}) a^\dagger(f_{2,t})|0\rangle. \tag{2.48}$$

Of course, one still has to show that the limit in fact exists, but this can be done by applying the techniques of subsection 2.2 once more.

If, instead of letting $t \to \infty$, one takes the limit $t \to -\infty$ in eq. (2.48), one obtains the in-going scattering state $|f_1, f_2 \text{ in}\rangle$. Finally, scattering states with sharp momenta may be defined through

$$|f_1, f_2 \text{ out}\rangle = \int_{-\pi}^{\pi} \Big(\prod_{i=1,2} \frac{d^3 p_i}{(2\pi)^3} \frac{\sqrt{Z(\mathbf{p}_i)}}{2\omega(\mathbf{p}_i)} f(\mathbf{p}_i) \Big) |\mathbf{p}_1, \mathbf{p}_2 \text{ out}\rangle \tag{2.49}$$

and the scattering matrix is given as usual by the scalar product of in- and out-going states. Note that the construction of the scattering states implies the free field normalization

$$\langle \mathbf{p}_1', \mathbf{p}_2' \text{ in}|\mathbf{p}_1, \mathbf{p}_2 \text{ in}\rangle = \langle \mathbf{p}_1'|\mathbf{p}_1\rangle\langle \mathbf{p}_2'|\mathbf{p}_2\rangle + \langle \mathbf{p}_1'|\mathbf{p}_2\rangle\langle \mathbf{p}_2'|\mathbf{p}_1\rangle \tag{2.50}$$

(and similarly for the out-states).

2.4. LSZ type formula for the scattering matrix

The scattering theory developed above is conceptually satisfactory and mathematically rigorous. However, the final expression (2.48) for the scattering states involves the Haag-Ruelle creation operators $a^\dagger(f)$ which are rather unwieldy objects from the point of view of the euclidean field theory we started off. More accessible are the euclidean correlation functions of the plaquette field $\mathcal{P}(x)$ and I would now like to show that the elastic scattering amplitude can actually be extracted in a comparatively simple way from the 4-point function of $\mathcal{P}(x)$.

In continuum quantum field theories there is a well-known and rigorous procedure (the "reduction" technique) which leads from the Haag-Ruelle construction of the scattering states to a formula for the scattering matrix in terms of the time ordered four-point function of the interpolating quantum field (e.g. [22], p.256 ff). Using the real-time operator field

$$\hat{\mathcal{P}}(t, \mathbf{x}) = e^{iHt}\, \hat{\mathcal{P}}(\mathbf{x})\, e^{-iHt}, \tag{2.51}$$

this procedure can be carried over to the lattice, although a rigorous proof of the existence of certain limits involved is not yet available. In

any case, I have little doubt that the resulting formulae are correct. Explicitly, they read

$$\langle \mathbf{p}_1, \mathbf{p}_2 \text{ out} | \mathbf{p}_3, \mathbf{p}_4 \text{ in} \rangle = \langle \mathbf{p}_1, \mathbf{p}_2 \text{ in} | \mathbf{p}_3, \mathbf{p}_4 \text{ in} \rangle + i(2\pi)^4 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\, \delta_P(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\, \mathcal{M}, \tag{2.52}$$

$$\mathcal{M} = \left\{ \left(\prod_{k=1}^{4} (E_k^2 - \omega_k^2)/\sqrt{Z_k} \right) \frac{1}{i} \tau_4(E_1, \mathbf{q}_1; \ldots; E_4, \mathbf{q}_4) \right\}_{\substack{E_i = \sigma_i \omega_i \\ \mathbf{q}_i = \sigma_i \mathbf{p}_i}}, \tag{2.53}$$

where I have introduced the abbreviations $\omega_k = \omega(\mathbf{p}_k)$, $Z_k = Z(\mathbf{p}_k)$ and

$$\sigma_i = \begin{cases} +1 & \text{if } i = 1, 2, \\ -1 & \text{if } i = 3, 4. \end{cases} \tag{2.54}$$

The function τ_4 is given by

$$(2\pi)^4 \delta(\sum_{k=1}^{4} E_k)\, \delta_P(\sum_{k=1}^{4} \mathbf{q}_k)\, \tau_4(E_1, \mathbf{q}_1; \ldots; E_4, \mathbf{q}_4) = \int \mathrm{d}t_1 \cdots \mathrm{d}t_4 \sum_{\mathbf{x}_1 \cdots \mathbf{x}_4} \exp\{i \sum_{k=1}^{4} (E_k t_k - \mathbf{q}_k \mathbf{x}_k)\} \times \langle 0 | \mathcal{T}\{\hat{\mathcal{P}}(t_1, \mathbf{x}_1) \cdots \hat{\mathcal{P}}(t_4, \mathbf{x}_4)\} | 0 \rangle^{\mathrm{con}} \tag{2.55}$$

($\mathcal{T}\{\cdots\}$ implies a time-ordered product such that the operators with the larger time arguments t_k come first).

Equation (2.53) is very similar to the well-known LSZ-formula for the scattering matrix in continuum quantum field theories, the obvious difference being that the wave function renormalization constant $Z(\mathbf{p})$ is momentum dependent in general and that the analytical form of $\omega(\mathbf{p})$ is not known a priori. Another, more subtle difference is that the τ_4-function (2.55) is not simply related to the euclidean correlation functions of the field $\mathcal{P}(x)$ through a Wick rotation in the complex energy plane. To understand what the relation is instead, it is helpful to first study the simpler case of the two-point function $\langle \mathcal{P}(x) \mathcal{P}(0) \rangle$.

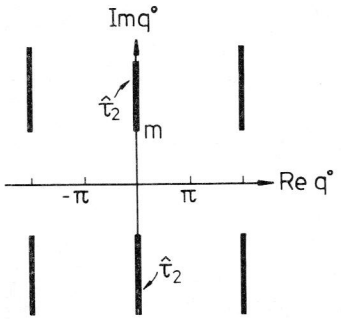


Fig. 4. Analyticity domain of $G_2(q)$ (fat lines represent the singular set $\mathcal{S}$). The function $\hat{\tau}_2$ is defined along the imaginary axis, approached from left or right depending on the sign of E.

Let $q = (q^0, \mathbf{q})$ be a euclidean four-momentum and consider the euclidean propagator

$$G_2(q) = \sum_x \mathrm{e}^{-iqx} \langle \mathcal{P}(x)\mathcal{P}(0) \rangle. \tag{2.56}$$

Using the spectral representation (2.23), we have

$$G_2(q) = \int_m^\infty \mathrm{d}E\, \rho(E, \mathbf{q}) \frac{\sinh E}{\cosh E - \cos q^0}. \tag{2.57}$$

At fixed $\mathbf{q}$, $G_2(q)$ thus extends to an analytic function of q^0 in the domain

$$\begin{aligned} &\mathcal{D}_2 = \mathrm{C} \setminus \mathcal{S}, \\ &\mathcal{S} = \{ q^0 \mid \mathrm{Re}\, q^0 = 0 \,(\mathrm{mod}\, 2\pi),\ |\mathrm{Im}\, q^0| \geq m \} \end{aligned} \tag{2.58}$$

(see fig. 4). In particular, a Wick rotation may be performed and one is thus led to define

$$\hat{\tau}_2(E, \mathbf{q}) = \frac{1}{i} G_2(q), \qquad q^0 = (i - \epsilon)E, \tag{2.59}$$

where E is real and $\epsilon > 0$ infinitesimal.

At first sight one might expect that $\hat{\tau}_2$ is equal to the Fourier transform

$$\tau_2(E,\mathbf{q}) = \int \mathrm{d}t \sum_{\mathbf{x}} \mathrm{e}^{i(Et-\mathbf{qx})} \langle 0|\mathcal{T}\{\hat{\mathcal{P}}(t,\mathbf{x})\hat{\mathcal{P}}(0,\mathbf{0})\}|0\rangle \tag{2.60}$$

of the real time propagator. This is however not the case: using the spectral representation (2.23) again, we have

$$\begin{aligned}&\langle 0|\mathcal{T}\{\hat{\mathcal{P}}(t,\mathbf{x})\hat{\mathcal{P}}(0,\mathbf{0})\}|0\rangle = \\ &\int_m^\infty \mathrm{d}E \int_{-\pi}^{\pi} \frac{\mathrm{d}^3q}{(2\pi)^3}\, \mathrm{e}^{i(\mathbf{qx}-E|t|)}\rho(E,\mathbf{q}),\end{aligned} \tag{2.61}$$

and hence

$$\tau_2(E,\mathbf{q}) = i\int_m^\infty \mathrm{d}E'\rho(E',\mathbf{q})\frac{2E'}{E^2-E'^2+i\epsilon}, \tag{2.62}$$

which looks quite different from

$$\hat{\tau}_2(E,\mathbf{q}) = i\int_m^\infty \mathrm{d}E'\rho(E',\mathbf{q})\frac{\sinh E'}{\cosh E-\cosh E'+i\epsilon}. \tag{2.63}$$

Still, an important observation is that the difference $\tau_2(E,\mathbf{q})-\hat{\tau}_2(E,\mathbf{q})$ extends to a holomorphic function of E in a strip around the real axis, as one may easily prove from eqs. (2.62),(2.63). In other words, τ_2 is equal to $\hat{\tau}_2$ up to a non-singular term.

The above considerations can be readily extended to the case of the four-point function. One first defines $G_4(q_1,\ldots,q_4)$ to be the Fourier transform of the connected correlation function $\langle \mathcal{P}(x_1)\cdots\mathcal{P}(x_4)\rangle^{\mathrm{con}}$. Then, using the spectral representation, one can show that G_4 is an analytic function of the energy components $q_1^0,\ldots,q_4^0$ in a domain $\mathcal{D}_4$ which is characterized by

$$\begin{aligned}&\sum_{k=1}^4 q_k^0 = 0,\\ &q_k^0 \notin \mathcal{S} \quad (\text{for all } k), \qquad q_k^0+q_j^0 \notin \mathcal{S} \quad (\text{for all } k\neq j).\end{aligned} \tag{2.64}$$

In particular, for real E_k with $\sum_{k=1}^4 E_k = 0$ one may define

$$\hat{\tau}_4(E_1,\mathbf{q}_1;\ldots;E_4,\mathbf{q}_4) = \frac{1}{i^3}G_4(q_1,\ldots,q_4), \qquad q_k^0 = (i-\epsilon)E_k, \tag{2.65}$$

and it is then possible to show that $\hat{\tau}_4$ differs from τ_4 only by terms which are regular in at least one of the variables E_k or $E_k + E_j$.

The import of this last remark is the following. Consider again the LSZ expression (2.53) for the elastic scattering amplitude which involves the function τ_4. In order that the matrix element $\mathcal{M}$ be non-zero, it is necessary that τ_4 has the appropriate on-shell singularities to cancel the factors $E_k^2 - \omega_k^2$ in the limit $E_k = \sigma_k \omega_k$. Since $\hat{\tau}_4$ differs from τ_4 only by terms which are regular in some combinations of the energy variables, the latter may be replaced by the first in eq. (2.53) without changing the result. Thus, the final formula for the elastic scattering amplitude $\mathcal{M}$, as defined through eq. (2.52), reads

$$\mathcal{M} = \left\{ \left(\prod_{k=1}^{4} (E_k^2 - \omega_k^2)/\sqrt{Z_k} \right) \frac{1}{i} \hat{\tau}_4(E_1, \mathbf{q}_1; \ldots; E_4, \mathbf{q}_4) \right\}_{\substack{E_i = \sigma_i \omega_i \\ \mathbf{q}_i = \sigma_i \mathbf{p}_i}} , \tag{2.66}$$

where the Wick rotated euclidean 4-point function $\hat{\tau}_4$ is given by eq. (2.65).

In infrared stable theories such as the φ^4-theory, $G_4(q_1, \ldots, q_4)$ can be calculated in perturbation theory and it is then easy to perform the Wick rotation and to determine the scattering matrix from eq. (2.66). The analytic continuation of the 4-point function should also not give rise to any great difficulties in the strong coupling expansion of lattice gauge theories, for example, but in a numerical simulation it seems hopeless to try to proceed via eq. (2.66), because G_4 is only approximately calculable by this method and only for a finite set of euclidean momenta. What would be needed is a formula which expresses the scattering matrix directly in terms of the position space correlation functions of the euclidean field $\mathcal{P}(x)$. Unfortunately, such a formula is not available presently, but an effort is being made to close this gap [24]. Meanwhile one may resort to a completely different method to compute scattering lengths (and perhaps other low energy scattering matrix elements) which is based on a set of relations between the volume dependence of the energy spectrum and certain elastic scattering processes in infinite volume. As has been demonstrated recently [48,49,53,54] finite size effects on the energy spectrum are relatively easy to observe in numerical simulations, at least when the particle interactions are sufficiently strong, and the prospects for this method are therefore rather good (see section 4 for further details).

3. Renormalization and continuum limit in perturbation theory

3.1. Examples and introductory remarks

As in continuum quantum field theories, the weak coupling expansion of the euclidean correlation functions in lattice field theories is obtained straightforwardly by expanding the functional integral about the classical minimum of the action. At any given order of the expansion, the correlation functions are equal to a sum of terms each of which can be graphically represented by a Feynman diagram. The corresponding Feynman rules are specific to the lattice action chosen and may be rather complicated, especially for gauge theories.

Consider for example the one-component φ^4 theory on a 4-dimensional hypercubic lattice with lattice spacing a. The action for this model is usually taken to be

$$S = a^4 \sum_x \left\{ \frac{1}{2} \sum_{\mu=0}^{3} (\partial_\mu \varphi(x))^2 + \frac{1}{2} m^2 \varphi(x)^2 + \frac{g}{4!} \varphi(x)^4 \right\},$$

$$\partial_\mu \varphi(x) = (\varphi(x + a\hat{\mu}) - \varphi(x))/a. \tag{3.1}$$

Thus, there is only one vertex in this case, viz.

$$= -g, \tag{3.2}$$

and the propagator is given by

$$= (\hat{p}^2 + m^2)^{-1},$$

$$\hat{p}_\mu = \frac{2}{a} \sin\left(\frac{1}{2} a p_\mu\right). \tag{3.3}$$

Note also that the integration range for internal lattice momenta is the Brillouin zone

$$\mathcal{B} = \{p \in \mathrm{R}^4 \mid |p_\mu| \leq \pi/a\}. \tag{3.4}$$

For $a > 0$ and $m^2 > 0$, the Feynman integrals are therefore absolutely convergent and the dependence on the external momenta is smooth.

In the limit $a \to 0$, the propagator (3.3) converges to the continuum expression $(p^2+m^2)^{-1}$ and the Brillouin zone becomes R^4, i.e. the lattice

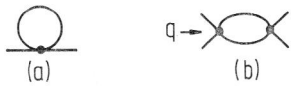


Fig. 5. Examples of one-loop diagrams in the φ^4 theory.

Feynman rules go over into the continuum rules. In particular, at tree-level of perturbation theory, the continuum limit exists in the expected way. At higher orders, the cutoff dependence of individual diagrams is singular in general and the renormalization procedure must be invoked before the continuum limit can be taken.

To illustrate the small a behaviour of Feynman integrals, consider the one-loop graphs shown in fig. 5. The diagram (a) is proportional to

$$J_1 = \int_B \frac{\mathrm{d}^4 k}{(2\pi)^4} (\hat{k}^2 + m^2)^{-1}. \tag{3.5}$$

Substituting $k \to k/a$ and using a Feynman parameter representation for the propagator, we have

$$\begin{aligned} J_1 &= \frac{1}{a^2} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-(am)^2 t} f(t), \\ f(t) &= \left[\mathrm{e}^{-2t} I_0(2t)\right]^4, \end{aligned} \tag{3.6}$$

where I_0 denotes a Bessel function. $f(t)$ is an analytic function with an asymptotic behaviour of the form

$$f(t) \underset{t\to\infty}{\sim} \frac{1}{(4\pi t)^2} \left\{1 + \sum_{\nu=1}^{\infty} c_\nu t^{-\nu}\right\}. \tag{3.7}$$

The integral over t in eq. (3.6) hence stays finite for $a \to 0$ and with a little more work it follows that

$$J_1 \underset{a\to 0}{\sim} \frac{1}{a^2} \left\{r_0 + (am)^2 \left(r_1 + s_1 \ln(am)^2\right) + \cdots\right\},$$

$$r_0 = 0.154933. \tag{3.8}$$

Thus, the diagram (a) has a quadratic and a subleading logarithmic divergence (they are both cancelled when the mass parameter m is renormalized).

The diagram (b) in fig. 5 depends on an external momentum q and is proportional to

$$J_2(q) = \int_{\mathcal{B}} \frac{\mathrm{d}^4k}{(2\pi)^4} \{(\hat{k}^2 + m^2)((\widehat{k+q})^2 + m^2)\}^{-1}. \tag{3.9}$$

Especially, for $q = 0$ we have

$$J_2(0) = -\frac{\partial}{\partial m^2} J_1, \tag{3.10}$$

and hence

$$J_2(0) = -s_1 \ln(am)^2 - r_1 - s_1 + O(a^2 \ln a^2). \tag{3.11}$$

The subtracted integral $J_2(q) - J_2(0)$, on the other hand, is convergent for $a \to 0$ as one may show using the Reisz power counting theorem which I will present later. As a result we have

$$J_2(q) \underset{a\to 0}{\sim} -s_1 \ln(am)^2 - r_1 - s_1 + \int \frac{\mathrm{d}^4k}{(2\pi)^4} (k^2 + m^2)^{-1} \times \{((k+q)^2 + m^2)^{-1} - (k^2 + m^2)^{-1}\} + \cdots, \tag{3.12}$$

which shows that the diagram (b) is logarithmically divergent in the continuum limit (the divergence is cancelled when the coupling g is renormalized).

For a general diagram D with L loops one expects that an asymptotic expansion of the form

$$D \underset{a\to 0}{\sim} a^{-\omega} \sum_{n=0}^{\infty} \sum_{l=0}^{L} c_{nl}\, a^n (\ln a)^l \tag{3.13}$$

holds, where $\omega \geq 0$ is an integer which depends on the convergence properties of D and of its sub-diagrams. Symanzik has given arguments for the validity of (3.13), along with an analysis of the structure of the

coefficients c_{nl}, but for $L \geq 2$ a completely rigorous proof is still missing [30-33].

As in other regularization schemes, the divergences of individual diagrams cancel in the sum of all diagrams of a given order provided the theory is properly renormalized. In the BPHZL approach to renormalization, one proceeds by the following three steps [26-29].

(1) To every diagram D a subtracted Feynman integral I_F is associated by applying a Taylor subtraction operator to the Feynman integrand. For the graph (b) discussed above, I_F is just equal to $J_2(q)-J_2(0)$, but in diagrams with more loops, subtractions are also performed for the subdiagrams in a nested way which is organized according to Zimmermann's famous forest formula.

(2) Using an appropriate version of the power counting theorem, the subtracted integral I_F is then shown to be absolutely convergent in the limit where the ultra-violet cutoff is removed.

(3) Finally, one proves that the subtractions made in step (1) can be effected by adding appropriate local interaction terms to the action. In renormalizable theories, the total number of such "counter-terms" is finite and their addition can usually be interpreted as a renormalization of the parameters in the original Lagrangian.

Steps (1) and (3) are of an essentially combinatoric nature and can therefore be carried over to the lattice without great difficulty [35,36]. An important difference is perhaps that the subtraction operators must be introduced in such a way that they correspond to local lattice counterterms, in particular, the periodicity in momentum space must be respected. The main obstacle for the realization of the BPHZL program on the lattice is that a power counting theorem for lattice Feynman diagrams must be found. Such a theorem has recently been established by Reisz and in what follows I would like to explain his result for the case where all propagators are massive.

3.2. Structure of lattice Feynman integrals

A Feynman integral on the lattice has the general form

$$I_F = \int_{\mathcal{B}} \mathrm{d}^4 k_1 \cdots \mathrm{d}^4 k_L \, V(k, q; m, a)/C(k, q; m, a), \tag{3.14}$$

where $k_1, \ldots, k_L$ are the loop momenta, $q_1, \ldots, q_E$ the external momenta and m collectively denotes the mass parameters of the theory. The

numerator V in eq. (3.14) includes all vertex factors and the numerators of the propagators, while C is the product of the denominators of the propagators.

The only properties V is assumed to have are the following.

V1. *There is an integer ω and a smooth function F such that*

$$V(k,q;m,a) = a^{-\omega} F(ak, aq; am). \tag{3.15}$$

Furthermore, F is 2π-periodic in the momenta ak_i and a polynomial in the masses am.

V2. *The continuum limit*

$$P(k,q;m) = \lim_{a\to 0} V(k,q;m,a) \tag{3.16}$$

exists.

Note that P is a homogeneous polynomial in k, q, m of degree ω (unless P vanishes identically). The requirements **V1** and **V2** are not very restrictive and are satisfied in all (local) lattice models I know of.

The denominator C in the Feynman integral (3.14) is assumed to have the structure

$$C(k,q;m,a) = \prod_{i=1}^{I} C_i(l_i;m,a), \tag{3.17}$$

where the line momenta $l_i(k,q)$ are linear combinations of $k_1,\ldots,k_L$ and $q_1,\ldots,q_E$. An example for an admissible propagator function C_i is $\hat{l}_i^2 + m_i^2$ with m_i being one of the mass parameters of the theory. In general, C_i is required to have the following properties [1].

C1. *There is a smooth function G_i such that*

$$C_i(l_i;m,a) = a^{-2} G_i(al_i; am). \tag{3.18}$$

Furthermore, G_i is 2π-periodic in the momentum al_i and a polynomial in the masses am.

C2. *The continuum limit of C_i exists and is given by*

$$\lim_{a\to 0} C_i(l_i;m,a) = l_i^2 + m_i^2, \tag{3.19}$$

[1] In his paper, Reisz makes stronger assumptions on the propagators than those listed here. For the proof of the power counting theorem, properties **C1**-**C3** are however all what is needed.

where $m_i > 0$ is a combination of the mass parameters m.

C3. *There are positive constants a_0, A such that*

$$|C_i(l_i; m, a)| \geq A(\hat{l}_i^2 + m_i^2) \tag{3.20}$$

for all $a \leq a_0$ and all l_i.

This last requirement is of a somewhat technical nature, but it is in general satisfied in purely bosonic theories and in models involving Wilson fermions, where the denominator of the propagator is proportional to

$$(1 + am)\hat{p}^2 + m^2 + \frac{1}{2}a^2 \sum_{\mu<\nu} \hat{p}_\mu^2 \hat{p}_\nu^2 \tag{3.21}$$

and thus satisfies (3.20) trivially. Staggered and naive fermions are excluded, however, because in these cases, the propagator functions C_i are of order 1 at some points at the boundary of the Brillouin zone while (3.20) requires them to be of order a^{-2} there.

For the validity of the power counting theorem, it is also necessary that the line momenta $l_i(k, q)$ are *natural.* This term refers to the following properties:

L1. *For all line momenta we have*

$$l_i(k, q) = \sum_{j=1}^{L} a_{ij} k_j + \sum_{l=1}^{E} b_{il} q_l \tag{3.22}$$

where $a_{ij} \in \mathrm{Z}$ and $b_{il} \in \mathrm{R}$.

L2. *Define the linear combinations*

$$p_i(k) = \sum_{j=1}^{L} a_{ij} k_j \tag{3.23}$$

and the set

$$\mathcal{L} = \{k_1, \ldots, k_L, p_1, \ldots, p_I\}. \tag{3.24}$$

Suppose $u_1, \ldots, u_L$ are L linearly independent elements of $\mathcal{L}$. Then we have

$$k_i = \sum_{j=1}^{L} c_{ij} u_j, \tag{3.25}$$

where $c_{ij} \in \mathbb{Z}$.

The emphasis here is on the requirement that the coefficients a_{ij} and c_{ij} be integers and not just any real numbers. Property **L1** guarantees that a shift of the integration variables k_j by an integer multiple of $2\pi/a$ results in a similar shift of the line momenta and hence leaves the Feynman integrand invariant. The second property **L2** says that one could choose some of the line momenta as new integration variables and still have the same integration domain and a periodic integrand. For integrals which are associated to a Feynman diagram, it is always possible to choose the loop momenta such that they coincide with some of the line momenta and **L1** and **L2** are then satisfied automatically.

3.3. Degrees of divergence

Power counting on the lattice requires the definition of a degree of divergence of the Feynman integrand relative to certain linear subspaces (Zimmermann subspaces) of the space of loop momenta. Since it is not entirely obvious how this degree should be introduced, it is useful to first study the general one-loop case ($L = 1$).

It is tempting to assume that a lattice Feynman integral is convergent for $a \to 0$ if the continuum limit of the integrand is absolutely integrable. However, this is not the case: the integrand of the integral

$$I_F = \int_{\mathcal{B}} \mathrm{d}^4k \sum_{\mu} (1 - \cos ak_\mu)/(\hat{k}^2 + m^2) \tag{3.26}$$

vanishes for $a = 0$ and is therefore integrable, but by substituting $k \to k/a$ one quickly sees that the integral is in fact quadratically divergent. Thus, a good definition of degree of divergence must not only refer to the properties of the Feynman integrand in the continuum limit. One rather needs a definition which characterizes the behavior of the integrand at small a and large k of order $1/a$.

With these considerations in mind, one is naturally led to the following definitions, valid for one-loop integrals. Let $\deg V$ be the integer ν which appears in the asymptotic expansion

$$V(\lambda k, q; m, a/\lambda) \underset{\lambda \to \infty}{=} K\lambda^\nu + O(\lambda^{\nu-1}), \qquad K \not\equiv 0 \tag{3.27}$$

($\nu = -\infty$ if the l.h.s. vanishes more rapidly than any power of λ^{-1}). The existence of the asymptotic expansion (3.27) is guaranteed by the

assumed properties of V. Next, one defines $\deg C$ in exactly the same way as $\deg V$ and then sets

$$\deg I_F = 4 + \deg V - \deg C. \tag{3.28}$$

The power counting theorem now simply states that the continuum limit of the integral I_F exists if $\deg I_F < 0$ and that it is given by

$$\lim_{a\to 0} I_F = \int \mathrm{d}^4k\, P(k,q;m) \Big/ \prod_{i=1}^{I} (l_i^2 + m_i^2), \tag{3.29}$$

where the integral on the r.h.s. is absolutely convergent. The proof of the theorem for the present case is actually rather easy and is therefore included in an appendix to this section.

To illustrate the theorem, consider again the subtracted one-loop integral $J_2(q) - J_2(0)$ (eq. (3.9)). In this case we have

$$\begin{aligned} V &= (\hat{k}^2 - (\widehat{k+q})^2)/(2\pi)^4, \\ C &= (\hat{k}^2 + m^2)^2((\widehat{k+q})^2 + m^2), \end{aligned} \tag{3.30}$$

and hence $\deg V = 1$, $\deg C = 6$. It follows that $\deg I_F = -1$ and the continuum limit of $J_2(q) - J_2(0)$ thus exists as anticipated.

I now proceed to discuss Feynman integrals with any number of loop momenta. Here it is not sufficient to consider the overall degree of divergence of the integral, but it is also necessary to study the behavior of the integrand when only some of the momenta k_i are large. This can be expressed more precisely by introducing the notion of a *Zimmermann subspace*. To this end consider again the set $\mathcal{L}$ of momenta defined through eqs. (3.22)-(3.24) and suppose

$$u_1, \ldots, u_d, v_1, \ldots, v_{L-d}, \qquad d \geq 1, \tag{3.31}$$

are L linearly independent elements of $\mathcal{L}$. By property **L2**, one could make a change variables and take $u_1, \ldots, v_{L-d}$ as the new integration variables. A $4d$-dimensional linear subspace H of the space of loop momenta k may now be defined by fixing $v_1, \ldots, v_{L-d}$. H is called a Zimmermann subspace where, by abuse of language, one does not distinguish between subspaces corresponding to different values of $v_1, \ldots, v_{L-d}$.

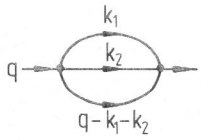


Fig. 6. Example of a two-loop graph. Here we have $\mathcal{L} = \{k_1, k_2, -k_1 - k_2\}$ and the non-trivial Zimmermann subspaces are obtained by fixing k_1, k_2 or $k_1 + k_2$.

Since $\mathcal{L}$ is a finite set, there is a finite number of choices for the basis (3.31) and hence only a finite number of Zimmermann subspaces. For example, for the graph shown in fig. 6 there are 4 of them.

Suppose now that H is a given Zimmermann subspace as described above. We then define an associated degree of divergence through

$$\deg_H I_F = 4d + \deg_H V - \deg_H C, \tag{3.32}$$

where $\deg_H V$ is equal to the integer ν in the asymptotic expansion

$$V(k(\lambda u, v), q; m, a/\lambda) \underset{\lambda\to\infty}{=} K\lambda^\nu + O(\lambda^{\nu-1}), \qquad K \not\equiv 0, \tag{3.33}$$

and $\deg_H C$ is defined similarly. Thus, $\deg_H I_F$ is just the degree which one would naturally associate with the integral when the momenta $v_1, \ldots, v_{L-d}$ are fixed and only the variables $u_1, \ldots, u_d$ are integrated over. In particular, for $L = 1$ there is only one Zimmermann subspace and $\deg_H I_F = \deg I_F$.

3.4. The Reisz power counting theorem

Having discussed all the notions needed, we can now write down the lattice power counting theorem established by Reisz [34].

Theorem. *Let I_F be a lattice Feynman integral with all the properties listed in subsection 3.2. Suppose in addition that $\deg_H I_F < 0$ for all Zimmermann subspaces H. Then, the continuum limit of I_F exists and is given by*

$$\lim_{a\to 0} I_F = \int \mathrm{d}^4 k_1 \cdots \mathrm{d}^4 k_L \, P(k, q; m) / \prod_{i=1}^{I} (l_i^2 + m_i^2), \tag{3.34}$$

where the integral on the r.h.s. is absolutely convergent.

This theorem can be considered the lattice version of the old power counting theorem of Hahn and Zimmermann [26] which applies to continuum Feynman integrals. Some important elements of the proof of the latter can actually be taken over to the present case, but the proof of Reisz' theorem is rather more complicated, mainly because the numerator $V(k, q; m, a)$ is allowed to be of a very general form. Thus, the reader is referred to the paper of Reisz for the proof while I proceed to discuss some of the implications and extensions of the theorem.

As explained in subsection 3.1, the BPHZ renormalization program can be carried over to the lattice, the key step being the lattice power counting theorem which is now at our disposal. As a result one obtains a proof of renormalizability to all orders of perturbation theory for a large class of lattice theories including the φ^4-theory and (massive) Yukawa theories with Wilson fermions [35]. An interesting aspect of this proof is that the addition of "irrelevant" terms to the action, such as

$$a^4 \sum_x \Big\{ a^2 \lambda \varphi^6 + a^4 \tau \sum_{\mu=0}^{4} (\partial_\mu \varphi)^4 \Big\} \tag{3.35}$$

in case of the φ^4-theory, has no influence on the renormalized perturbation expansion in the continuum limit. In other words, the continuum limit is *universal* in perturbation theory, i.e. it does not depend on how exactly one chooses the lattice action as long as it is local and has the correct classical continuum limit.

With the obvious modifications, the lattice power counting theorem also holds on hypercubic lattices of arbitrary dimensionality. A nice application of the theorem in two dimensions has recently been described by Golterman and Petcher [38]. They showed that the Wess-Zumino model, appropriately latticised, indeed becomes super-symmetric in the continuum limit, to all orders of perturbation theory and without fine-tuning of the parameters in the action.

So far I have discussed Feynman diagrams with massive propagators only. The power counting theorem can however be extended to the case where some or all of the masses vanish [36]. One then also has to worry about the convergence of the Feynman integrals for $a > 0$, because massless propagators are singular at zero momentum and a product of them may not be integrable. This problem is solved by introducing appropriate infrared degrees of divergence and including an associated convergence condition in the formulation of the power counting theorem.

An important application of the power counting theorem for Feynman diagrams with massless propagators is the proof of renormalizability, to all orders in perturbation theory, of pure non-Abelian lattice gauge theories with e.g. the Wilson plaquette action. To anybody who has performed explicit perturbative calculations in this model it must be obvious that this recent result of Reisz [37] is very non-trivial. The difficulty is not only that the Feynman rules involve complicated vertices of arbitrary order, but one also has to control the structure of the counter-terms to such an extent that they can be shown to amount to just a renormalization of the gauge coupling g_0 and of the normalizations of the gauge and Fadeev–Popov fields. As in the continuum, the solution to this problem is based on an analysis of the Ward identities which are associated to the Becchi-Rouet-Stora (BRS) symmetry of the gauge-fixed action [39,40]. That the BRS transformation is also an exact symmetry on the lattice may seem a little surprising, but as I shall explain below, the existence of the BRS symmetry is essentially a group theoretical lemma which does not in any way refer to the underlying space-time structure [42].

3.5. Gauge fixing and BRS symmetry on the lattice

In lattice gauge theories, such as for example the pure glue theories discussed in subsection 2.1, the perturbation expansion is obtained by first fixing the gauge degrees of freedom and then performing a saddle point expansion about the absolute minimum of the total action

$$S_{\text{total}} = S + S_{\text{gf}} + S_{\text{FP}}, \tag{3.36}$$

where S denotes the original action, S_{gf} the gauge fixing term and S_{FP} the Fadeev–Popov ghost action. I shall later write down the ghost action explicitly for a simple choice of the gauge fixing condition. It is possible to show from that expression that the total action has a BRS symmetry [41], but due to the lattice complications, the proof is lengthy and not very illuminating. Instead, I shall proceed on a slightly more abstract level and the BRS symmetry will then easily be seen to be a general property of the gauge fixing procedure (for a different, although ultimately equivalent point of view, see [42])

It is well known that common gauge fixing conditions are stricken with the Gribov copy problem – on a given gauge orbit there may be several field configurations satisfying the gauge condition – and this is certainly one of the reasons why, in non-perturbative calculations, gauge fixing

is unpopular. The problem originates in the complicated, "non-linear" geometry of the gauge group and the space of gauge fields. It can in principle be avoided by splitting the configuration space in a gauge invariant manner into a number of disjoint sets and by employing different gauge fixing conditions in these patches. For perturbation theory, gauge fixing is, however, only needed in an infinitesimal neighborhood of the classical vacuum configurations, because any field configuration which is a finite distance away from the pure gauge orbit makes an exponentially small contribution to the functional integral. As I shall explain in more detail below, there are no topological obstructions against choosing any particular (reasonable) gauge in a finite but sufficiently small neighborhood of the classical vacuum and thus I shall restrict myself to this situation in what follows.

As a concrete example, consider again the pure SU(N) gauge model introduced in subsection 2.1 [2]. Because perturbation theory and the gauge fixing procedure are unduly complicated in a gauge theory enclosed in a periodic box (see e.g. refs.[43,62]), I here choose Dirichlet boundary conditions instead to be able to bring out the essential points more clearly. In any case, the choice of boundary conditions should not matter in the infinite volume limit.

Thus, the space-time lattice Λ is here taken to be the cube

$$\Lambda = \left\{ x \in \mathrm{Z}^4 \;\middle|\; -L/2 < x_\mu \leq L/2 \;\; (\mu = 0, \ldots, 3) \right\}, \tag{3.37}$$

where L is an integer, the lattice size. The link variables $U(x, \mu)$ are defined when $x \in \Lambda$ and $x + \hat{\mu} \in \Lambda$, and the sum over plaquettes in the definition (2.2) of the action $S[U]$ is to be restricted accordingly. Dirichlet boundary conditions mean that we impose

$$U(x, \mu) = 1 \quad \text{if } x \in \partial\Lambda \text{ and } x + \hat{\mu} \in \partial\Lambda, \tag{3.38}$$

where $\partial\Lambda$ denotes the boundary of Λ. The set $\mathcal{F}$ of all such fields is isomorphic to a (huge) power of SU(N) and is hence a compact differentiable manifold.

The elements $g(x)$, $x \in \Lambda$, of the gauge group $\mathcal{G}$ act on the gauge fields $U \in \mathcal{F}$ according to eq. (2.4). In order to preserve the boundary condition (3.38), we require

$$g(x) = 1 \quad \text{for all } x \in \partial\Lambda. \tag{3.39}$$

[2] In particular, I again set the lattice spacing a equal to 1 in this subsection.

Actually, the boundary conditions on U are preserved by all transformations g which are constant along $\partial\Lambda$, but it is only the restricted set (3.39) of gauge transformations which will be fixed and the constant gauge transformations then survive as a global symmetry of the system. Thus, in what follows $\mathcal{G}$ denotes the set of all gauge transformations g with the boundary condition (3.39) and it is this set which will be referred to as the gauge group.

Obviously, $\mathcal{G}$ is a compact Lie group which acts in a differentiable manner on the field manifold $\mathcal{F}$, viz.

$$g \in \mathcal{G}, U \in \mathcal{F} \mapsto g \cdot U = U^g \in \mathcal{F}. \tag{3.40}$$

This action has the elementary properties

$$g_1 \cdot (g_2 \cdot U) = (g_1 g_2) \cdot U, \tag{3.41}$$

$$1 \cdot U = U. \tag{3.42}$$

A less trivial observation is that $\mathcal{G}$ acts *freely* on $\mathcal{F}$, i.e. $g \cdot U = U$ for some U implies $g = 1$. Indeed, this can quickly be shown by solving the equation $g \cdot U = U$ recursively for $g(x)$ starting at the boundary $\partial\Lambda$ where $g(x)$ is known to be equal to 1. An important consequence of this property of the gauge transformations is that the orbit manifold $\mathcal{M} = \mathcal{F}/\mathcal{G}$ is a differentiable manifold and that moreover the field space $\mathcal{F}$ can be regarded as a principal $\mathcal{G}$-bundle over $\mathcal{M}$. Actually, in the case at hand, this general statement is trivial to verify: in every gauge orbit there is a unique element $\hat{U}$ with

$$\hat{U}(x,0) = 1 \quad \text{for all } x \in \Lambda \text{ with } -L/2 < x_0 \leq L/2 - 2, \tag{3.43}$$

and any other element U in the orbit can be written uniquely as $U = g \cdot \hat{U}$, where $g \in \mathcal{G}$. In other words, $\mathcal{M}$ is isomorphic to the manifold of all fields $\hat{U} \in \mathcal{F}$ with the above property and

$$\mathcal{F} = \mathcal{G} \times \mathcal{M}, \tag{3.44}$$

i.e. $\mathcal{F}$ is a trivial bundle over $\mathcal{M}$.

From now on I will proceed on a somewhat more abstract level and just assume the space $\mathcal{F}$ and the goup $\mathcal{G}$ have the general properties mentioned above, but are otherwise arbitrary. We are then interested in integrals of the form

$$I = \int_{\mathcal{F}} \mathrm{d}\mu(U)\, f(U), \tag{3.45}$$

where $d\mu(U)$ is some measure on $\mathcal{F}$ and $f(U)$ some integrable function, both being invariant under the action of $\mathcal{G}$. More specifically, I will only consider the case where $f(U)$ vanishes outside a small neighborhood of the orbit $[U_0]$ passing through a point $U_0 \in \mathcal{F}$, which I will refer to as the vacuum configuration. As discussed above, for the application to perturbation theory, it is sufficient to study this case.

Before I can explain how the gauge fixing is done in the integral (3.45), I need to introduce some further notation. Let T^a, $a = 1, \ldots, d_{\mathcal{G}}$, be a basis of the Lie algebra $\mathcal{L}_{\mathcal{G}}$ of $\mathcal{G}$, and $X = X_a T^a$ a general element of $\mathcal{L}_{\mathcal{G}}$ (in what follows, I use the summation convention for repeated basis labels $a, b, \ldots$). For any differentiable function $F(U)$ on $\mathcal{F}$, define

$$\delta_X F(U) = X_a \left\{ \frac{\partial}{\partial Y_a} F(\mathrm{e}^{-Y} \cdot U) \right\}_{Y=0} . \tag{3.46}$$

By construction, δ_X is a first order differential operator, linear in X, which satisfies

$$[\delta_X, \delta_Y] = \delta_{[X,Y]} \tag{3.47}$$

as a consequence of the group composition law (3.41). Next, let $\mathcal{N}$ be any subset of $\mathcal{F}$. We then define

$$[\mathcal{N}] = \left\{ U \in \mathcal{F} \,\middle|\, g \cdot U \in \mathcal{N} \text{ for some } g \in \mathcal{G} \right\}, \tag{3.48}$$

which is just the union of all gauge orbits passing through $\mathcal{N}$.

Gauge fixing begins by introducing a function F (the "gauge fixing condition") with the following properties.

F1. *F is a differentiable mapping from an open neighborhood $\mathcal{N} \subset \mathcal{F}$ of the vacuum configuration U_0 to the Lie algebra $\mathcal{L}_{\mathcal{G}}$ of $\mathcal{G}$.*

F2. *For any given $U \in [\mathcal{N}]$, there exists a unique gauge transformation g such that $g \cdot U \in \mathcal{N}$ and $F(g \cdot U) = 0$. Furthermore, $F(U_0) = 0$.*

F3. *For all $U \in \mathcal{N}$, we have $\det L(U) \neq 0$, where $L(U) : \mathcal{L}_{\mathcal{G}} \mapsto \mathcal{L}_{\mathcal{G}}$ is the linear operator defined through*

$$L(U) \cdot X = \delta_X F(U) \quad \text{for all } X \in \mathcal{L}_{\mathcal{G}}. \tag{3.49}$$

These properties imply that the solutions to the equation $F(U) = 0$, $U \in \mathcal{N}$, form a smooth submanifold which meets any gauge orbit passing through $\mathcal{N}$ at exactly one point. Note that $F(U)$ is not required to be defined far away from the vacuum configuration U_0, since ultimatly

we are only interested in perturbation theory and for this purpose it is sufficient to fix the gauge locally around U_0 (as we shall see). From a mathematical point of view, the restriction to a neighborhood of U_0 implies the absence of topological obstructions, i.e. gauge fixing functions always exist and in the concrete case of the SU(N) gauge theory examples are easy to find.

Given a gauge fixing function F, the fundamental statement now is the following

Lemma. *For any function $f(U)$ supported in $[\mathcal{N}]$, we have*

$$\int_{\mathcal{F}} \mathrm{d}\mu(U) f(U) = K \int_{\mathcal{N}} \mathrm{d}\mu(U) f(U) \det L(U)\, \delta(F(U)), \tag{3.50}$$

where K is a constant independent of f.

Proof: Let $\mathrm{d}g$ denote the Haar measure on $\mathcal{G}$ and set

$$\chi(U) = \begin{cases} 1 & \text{if } U \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.51}$$

Since $\mathrm{d}\mu(U)$ and $f(U)$ are gauge invariant, the r.h.s. in eq. (3.50) is proportional to

$$\int_{\mathcal{F}} \mathrm{d}\mu(U) f(U) \int_{\mathcal{G}} \mathrm{d}g\, \chi(g \cdot U) \det L(g \cdot U)\, \delta(F(g \cdot U)), \tag{3.52}$$

where $\mathrm{d}g$ denotes the invariant measure on $\mathcal{G}$. By assumption, $f(U)$ vanishes when $U \notin [\mathcal{N}]$. On the other hand, when $U \in [\mathcal{N}]$, the inner integral in eq. (3.52) can be evaluated as follows. According to property **F2**, there is a unique $h \in \mathcal{G}$ such that $h \cdot U \in \mathcal{N}$ and $F(h \cdot U) = 0$. Thus, the integral over g only receives contributions from an infinitesimal neighborhood of h. We may therefore substitute

$$g = \mathrm{e}^{-X} h, \tag{3.53}$$

and use $X \in \mathcal{L}_{\mathcal{G}}$ as the new integration variable. Noting

$$\begin{aligned} \mathrm{d}g &= \mathrm{d}X\{1 + O(X)\}, \\ F(g \cdot U) &= L(h \cdot U) \cdot X + O(X^2), \end{aligned} \tag{3.54}$$

where $\mathrm{d}X$ is the usual translation invariant measure on $\mathcal{L}_{\mathcal{G}}$, the inner integral in eq. (3.52) is seen to be equal to the sign of $\det L(h \cdot U)$. By property **F3**, this is independent of U and the Lemma hence follows. ∎

The expansion of the gauge fixed integral (3.50) in powers of the bare gauge coupling g_0 would be straightforward, but for the discussion of renormalization it is actually advantageous to first pass to yet another form of the integral in which the delta function constraint is replaced by an additon $S_{\rm gf}[U]$ to the original action. To this end, choose a positive definite and gauge invariant scalar product (X, Y) on the Lie algebra $\mathcal{L}_\mathcal{G}$. Then, it is quite obvious (and one can show rigorously) that for all $Z \in \mathcal{L}_\mathcal{G}$ with say $(Z, Z) < \epsilon$, the solutions to the equation $F(U) = Z$, $U \in \mathcal{N}$, form a smooth submanifold which meets any gauge orbit sufficiently close to the vacuum orbit $[U_0]$ at exactly one point. With no loss of generality, we may assume that this is true for all orbits passing through $\mathcal{N}$ (if not, make $\mathcal{N}$ smaller). The Lemma then holds unchanged when the delta function in eq. (3.50) is replaced by $\delta\big(F(U) - Z\big)$. Since the integral on the left of eq. (3.50) is independent of Z, we can take the average over all Z with $(Z, Z) < \epsilon$, using a Gaussian weight function, and as a result one obtains

$$\int_\mathcal{F} \mathrm{d}\mu(U) f(U) = K_\epsilon \int_{\mathcal{N}_\epsilon} \mathrm{d}\mu(U)\, f(U) \det L(U)\, \mathrm{e}^{-S_{\rm gf}[U]}. \tag{3.55}$$

Here, K_ϵ is a new constant,

$$\mathcal{N}_\epsilon = \big\{U \in \mathcal{N} \;\big|\; \big(F(U), F(U)\big) < \epsilon\big\} \tag{3.56}$$

another open neighborhood of U_0, and

$$S_{\rm gf}[U] = \frac{\lambda_0}{2g_0^2}\big(F(U), F(U)\big). \tag{3.57}$$

For later convenience, I have included a gauge parameter λ_0 and the gauge coupling g_0 in the definition of $S_{\rm gf}$.

A crucial observation now is that the gauge fixed integral (3.55) has a BRS symmetry which has exactly the same algebraic form (and also the same uses) as in continuum gauge theories. To exhibit the symmetry, we introduce Fadeev–Popov ghosts c and $\bar{c}$ to lift the determinant in eq. (3.55) to the exponent as usual. c and $\bar{c}$ are anti-commuting variables taking values in the Lie algebra $\mathcal{L}_\mathcal{G}$, viz.

$$c = c_a T^a, \qquad \bar{c} = \bar{c}_a T^a, \tag{3.58}$$

$$[c_a, c_b]_+ = [\bar{c}_a, \bar{c}_b]_+ = [c_a, \bar{c}_b]_+ = 0. \tag{3.59}$$

The integral on the r.h.s. of eq. (3.55) may then be written as

$$\int_{\mathcal{N}_c} \mathrm{d}\mu(U)\,\mathrm{d}c\,\mathrm{d}\bar{c}\,f(U)\exp\{-S_{\mathrm{gf}}[U]-S_{\mathrm{FP}}[U,c,\bar{c}]\} \tag{3.60}$$

where

$$S_{\mathrm{FP}}[U,c,\bar{c}]=(\bar{c},L(U)\cdot c)\,. \tag{3.61}$$

The claim now is that the BRS transformation

$$U\to \mathrm{e}^{\varepsilon c}\cdot U, \tag{3.62}$$

$$c\to c+\varepsilon\frac{1}{2}c_a c_b\left[T^a,T^b\right], \tag{3.63}$$

$$\bar{c}\to\bar{c}+\varepsilon\frac{\lambda_0}{g_0^2}F(U), \tag{3.64}$$

where ε is an infinitesimal anti-commuting parameter, leaves the action $S_{\mathrm{gf}}+S_{\mathrm{FP}}$ as well as the measure $\mathrm{d}\mu(U)\,\mathrm{d}c\,\mathrm{d}\bar{c}$ invariant.

The proof of this statement is simple. First note that the BRS variation of S_{gf} is given by

$$\begin{aligned}\delta S_{\mathrm{gf}}&=-\frac{\lambda_0}{g_0^2}\left(F(U),\delta_{\varepsilon c}F(U)\right)\\&=-\frac{\lambda_0}{g_0^2}\varepsilon\left(F(U),L(U)\cdot c\right)\end{aligned} \tag{3.65}$$

As a result we have

$$\delta S_{\mathrm{gf}}+\delta S_{\mathrm{FP}}=-\left(\bar{c},\delta_{\varepsilon c}L(U)\cdot c\right)+\left(\bar{c},L(U)\cdot\delta c\right). \tag{3.66}$$

Next, we note

$$\begin{aligned}\delta_{\varepsilon c}L(U)\cdot c&=\varepsilon c_a c_b\delta_{T^a}\delta_{T^b}F(U)\\&=\frac{1}{2}\varepsilon c_a c_b\delta_{[T^a,T^b]}F(U)\\&=L(U)\cdot\frac{1}{2}\varepsilon c_a c_b\left[T^a,T^b\right],\end{aligned} \tag{3.67}$$

and the two terms in eq. (3.66) thus cancel giving $\delta(S_{\mathrm{gf}}+S_{\mathrm{FP}})=0$ as asserted. I would like to emphasize that in the second step in eq. (3.67)

we have made use of the representation property (3.47) of the differential operators δ_X, which is quite crucial, since otherwise one would not know how to get rid of the second order derivatives on $F(U)$.

The BRS invariance of the measure $d\mu[U]$ follows from the observation that the transformation (3.62) is just an infinitesimal gauge transformation. The invariance of the ghost measure dc, on the other hand, is due to the fact that the group structure constants are traceless (in any basis), while $d\bar{c}$ is trivially invariant, because $\bar{c}$ is only shifted by an amount independent of $\bar{c}$.

In the way I have presented the BRS symmetry, it is obvious that it is a very general feature of the gauge fixing procedure, in particular, there is no reference to the underlying space-time manifold. Whether it is a lattice, a continuum or even not present at all, is therefore of no import. Moreover, the gauge fixing function $F(U)$ can be quite arbitrary, as only the properties **F1**–**F3** must be guaranteed.

I now proceed to show how the general formalism of gauge fixing and the BRS symmetry translates into concrete expressions for the case of the pure SU(N) gauge theory discussed at the beginning of this section. For the vacuum configuration U_0, we make the obvious choice

$$U_0(x,\mu) = 1 \quad \text{for all } x, \mu. \tag{3.68}$$

In a sufficiently small neighborhood of this configuration any field may be represented uniquely through

$$\begin{aligned} U(x,\mu) &= \exp g_0 A_\mu(x), \\ A_\mu(x) &= A_\mu^\alpha(x)\lambda^\alpha, \end{aligned} \tag{3.69}$$

where $\lambda^\alpha, \alpha = 1, \ldots, N^2-1$, denote the (anti-hermitian) generators of SU(N) and the gauge potential $A_\mu(x)$ is small, say $\left|g_0 A_\mu^\alpha(x)\right| < \epsilon$. The boundary conditions on $U(x,\mu)$ imply

$$A_\mu(x) = 0 \quad \text{if } x \in \partial\Lambda \text{ and } x + \hat{\mu} \in \partial\Lambda. \tag{3.70}$$

Infinitesimal gauge transformations $g(x)$ have the form

$$g(x) = 1 + \omega X(x), \tag{3.71}$$

where $X(x)$ is a field on Λ taking values in the Lie algebra of SU(N), i.e. at every lattice point, $X(x)$ is a traceless anti-hermitian $N \times N$-matrix,

and ω is an infinitesimal parameter. The boundary condition on $X(x)$ is

$$X(x) = 0 \quad \text{if } x \in \partial\Lambda. \tag{3.72}$$

Thus, the Lie algebra $\mathcal{L}_\mathcal{G}$ of the gauge group $\mathcal{G}$ can identified with the set of all such fields $X(x)$, with the obvious commutator rule. A gauge invariant scalar product on $\mathcal{L}_\mathcal{G}$ is then given by

$$(X, Y) = -2 \sum_{x \in \Lambda} \mathrm{tr}\{X(x)Y(x)\}, \tag{3.73}$$

where "tr" means the ordinary matrix trace.

Next, consider the differential operators δ_X defined by eq. (3.46). If we choose $F(U) = A_\mu(x)$ in that equation, with $A_\mu(x)$ given by eq. (3.69), we have (after some algebra)

$$\delta_X A_\mu(x) = \frac{\mathrm{Ad}A_\mu}{1 - \exp[-g_0 \mathrm{Ad}A_\mu(x)]} \partial_\mu X(x) + \mathrm{Ad}A_\mu(x)X(x). \tag{3.74}$$

I here use the symbol "Ad" to denote the adjoint action of the Lie algebra of SU(N) on itself, i.e.

$$\mathrm{Ad}A_\mu(x)X(x) = [A_\mu(x), X(x)], \tag{3.75}$$

and the derivative appearing in eq. (3.74) is the lattice derivative (3.1). From the general theory, we have

$$\delta_X(\delta_Y A_\mu) - \delta_Y(\delta_X A_\mu) = \delta_{[X,Y]} A_\mu, \tag{3.76}$$

a relation, which would be rather difficult to prove from the explicit expression (3.74).

Now we choose the gauge fixing function $F(U)$ to be

$$F(U)(x) = \begin{cases} g_0 \sum_{\mu=0}^{3} \partial_\mu^* A_\mu(x) & \text{if } x \in \Lambda \setminus \partial\Lambda, \\ 0 & \text{if } x \in \partial\Lambda, \end{cases} \tag{3.77}$$

where ∂_μ^* is the adjoint lattice derivative, viz.

$$\partial_\mu^* f(x) = f(x) - f(x - \hat{\mu}). \tag{3.78}$$

The coupling g_0 is included in the definition of F to make sure that F is a function of U only and not a function of U and g_0. Note that F is

only well defined in an neighborhood of U_0 where the parametrization (3.69) is valid. It is possible to show that this gauge fixing function has all the required properties **F1–F3** for a sufficiently small neighborhood $\mathcal{N}$ of U_0 independent of g_0.

The explicit expressions for $S_{\rm gf}$ and $S_{\rm FP}$ are now easily obtained by inserting the above definitions in eqs. (3.49),(3.57) and (3.61). The result is

$$S_{\rm gf}[U] = -\sum_{x\in\Lambda}\sum_{\mu,\nu=0}^{3} \lambda_0 \,\mathrm{tr}\{\partial^*_\mu A_\mu(x)\partial^*_\nu A_\nu(x)\}, \tag{3.79}$$

$$S_{\rm FP}[U,c,\bar{c}] = \sum_{x\in\Lambda} \mathrm{tr}\{\bar{c}\,\Delta_{\rm FP} c\}, \tag{3.80}$$

$$\Delta_{\rm FP} = \sum_{\mu=0}^{3} \partial^*_\mu \left\{ \frac{g_0 \mathrm{Ad}A_\mu(x)}{1-\exp[-g_0\mathrm{Ad}A_\mu(x)]}\partial_\mu + g_0\mathrm{Ad}A_\mu(x)\right\} \tag{3.81}$$

(the Fadeev–Popov ghost fields $c(x)$ and $\bar{c}(x)$ take values in the Lie algebra of SU(N)). From here on the expansion of the functional integral in powers of g_0 is straightforward although algebraically complicated due to the lattice artifacts. An important point to note is that the restriction of the integral (3.60) to a small neighborhood of the vacuum configuration U_0 is of no concern to perturbation theory, because this restriction is only felt at values of A_μ of order $1/g_0$ and hence would only influence the calculation on a non-perturbative level.

Finally, the BRS symmetry assumes the form

$$\delta A_\mu(x) = -g_0\delta_{\varepsilon c} A_\mu(x), \tag{3.82}$$

$$\delta c(x) = \varepsilon g_0 c(x)c(x), \tag{3.83}$$

$$\delta\bar{c}(x) = \varepsilon\lambda_0 \sum_{\mu=0}^{3} \partial^*_\mu A_\mu(x), \tag{3.84}$$

which looks very much the same as the transformation in continuum gauge theories, except of course that the gauge variation $\delta_{\varepsilon c}A_\mu$ is more complicated on the lattice (cf. eq. (3.74)).

Appendix 1. Proof of the lattice power counting theorem for $L = 1$

The idea of proof is the following. Let

$$\chi(k) = \begin{cases} 1 & \text{if } k \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.85}$$

be the characteristic function of the Brillouin zone and define

$$f(k, q; m, a) = \chi(k) V(k, q; m, a) / C(k, q; m, a). \tag{3.86}$$

Now suppose we can find a function $g(k, q; m) \geq 0$ which is integrable, viz.

$$\int_{-\infty}^{\infty} \mathrm{d}^4 k \, g(k, q; m) < \infty, \tag{3.87}$$

and which satisfies

$$|f(k, q; m, a)| \leq g(k, q; m) \tag{3.88}$$

for all $k \in \mathrm{R}^4$ and all $a \leq \varepsilon$, where ε is some positive number (q and m are kept fixed and are thus regarded as constants in what follows). By the dominated convergence theorem of Lebesgue, we would then conclude that

$$\begin{aligned} \lim_{a \to 0} I_F &= \lim_{a \to 0} \int_{-\infty}^{\infty} \mathrm{d}^4 k \, f(k, q; m, a) \\ &= \int_{-\infty}^{\infty} \mathrm{d}^4 k \lim_{a \to 0} f(k, q; m, a) \end{aligned} \tag{3.89}$$

and since

$$\lim_{a \to 0} f(k, q; m, a) = P(k, q; m) \Big/ \prod_{i=1}^{I} (l_i^2 + m_i^2) \tag{3.90}$$

pointwise, the theorem follows.

Thus, we only need to show that a function $g(k, q; m)$ with the properties listed above exists. This will be achieved by establishing the bounds [1]

$$|C(k, q; m, a)| \geq \alpha (k^2 + \beta)^{\frac{1}{2} \deg C}, \tag{3.91}$$

[1] If $\deg V = -\infty$, the inequality (3.92) holds with $\deg V$ replaced by any arbitrarily large negative integer.

$$|V(k,q;m,a)| \leq \gamma(k^2+\delta)^{\frac{1}{2}\deg V}, \tag{3.92}$$

for all $k \in \mathcal{B}$ and sufficiently small a (i.e. $a \leq \varepsilon$). Here and below, α, β, ... generically denote some positive constants. Given (3.91) and (3.92), the choice

$$g(k,q;m) = \gamma(k^2+\delta)^{\frac{1}{2}\deg V}/\alpha(k^2+\beta)^{\frac{1}{2}\deg C} \tag{3.93}$$

has all the required properties, in particular, g is integrable since we have assumed that $\deg I_F = 4 + \deg V - \deg C$ is negative.

To prove the bound (3.91), we first note that the naturalness of the line momenta (assumptions **L1**, **L2**) implies

$$l_i = a_i k + \sum_{l=1}^{E} b_{il} q_l, \tag{3.94}$$

where $a_i \in \{1, 0, -1\}$. It follows that for $k \in \mathcal{B}$, the line momenta are either inside $\mathcal{B}$ or at most a distance of O(1) away from $\mathcal{B}$. In other words, we may choose $\varepsilon > 0$ such that for $a \leq \varepsilon$, $k \in \mathcal{B}$ and $i = 1, \ldots, I$ we have

$$\frac{3}{4} l_i \in \mathcal{B}. \tag{3.95}$$

We may also assume that $\varepsilon \leq a_0$ so that the bound (3.20) on the propagator functions C_i applies. Now it is trivial to show that for line momenta l_i satisfying (3.95), we have

$$\hat{l}_i^2 \geq \alpha l_i^2. \tag{3.96}$$

Hence, using (3.20), it follows that

$$|C(k,q;m,a)| \geq \beta \prod_{i=1}^{I} (\alpha l_i^2 + m_i^2) \tag{3.97}$$

for all $k \in \mathcal{B}$ and all $a \leq \varepsilon$.

In view of eq. (3.94), the factors in the above product can be bounded by

$$\alpha l_i^2 + m_i^2 \geq \begin{cases} \gamma k^2 + \delta & \text{if } |a_i| = 1, \\ \delta & \text{if } a_i = 0. \end{cases} \tag{3.98}$$

As result one obtains (with new constants α, β)

$$|C(k,q;m,a)| \geq \alpha(k^2+\beta)^{d_C}, \tag{3.99}$$

$$d_C = \sum_{i=1}^{I} |a_i| \,. \tag{3.100}$$

Finally, from the definition of $\deg C$ it is not difficult to deduce that $\deg C = 2d_C$ and the bound (3.91) thus follows.

To establish the bound (3.92) on the numerator V of the Feynman integrand, we start from eq. (3.15) which I here write in the form

$$V(k,q;m,a) = a^{-\omega} F(u,v), \tag{3.101}$$

where $u = ak$ and $v = (aq, am)$ ($u \in \mathrm{R}^4$, $v \in \mathrm{R}^n$ for some n). The definition (3.27) of $\nu = \deg V$ then implies

$$F(u,sv) \underset{s\to 0}{=} O(s^\tau), \qquad \tau = \omega - \nu \geq 0. \tag{3.102}$$

This means that the first $\tau - 1$ derivatives of $F(u,v)$ with respect to v vanish at $v = 0$. Recalling Taylor's formula

$$\begin{aligned} \phi(x+h) &= \sum_{r=0}^{N} \frac{1}{r!} (h\cdot\partial)^r \phi(x) + R_N, \\ R_N &= \frac{1}{N!} \int_0^1 \mathrm{d}t\, (1-t)^N (h\cdot\partial)^{N+1} \phi(x+th), \end{aligned} \tag{3.103}$$

it follows that for $\tau \geq 1$ we have

$$F(u,v) = \sum_{\mu_1=1}^{n} \cdots \sum_{\mu_\tau=1}^{n} v_{\mu_1} \cdots v_{\mu_\tau} H_{\mu_1\cdots\mu_\tau}(u,v), \tag{3.104}$$

where $H_{\mu_1\cdots\mu_\tau}(u,v)$ are smooth functions which are explicitly given by

$$\begin{aligned} H_{\mu_1\cdots\mu_\tau}(u,v) = \frac{1}{(\tau-1)!} \int_0^1 \mathrm{d}t\, (1-t)^{\tau-1} \\ \times \frac{\partial}{\partial w_{\mu_1}} \cdots \frac{\partial}{\partial w_{\mu_\tau}} F(u, w+tv)\Big|_{w=0} . \end{aligned} \tag{3.105}$$

Now we distinguish two cases according to whether $\nu \leq 0$ or $\nu > 0$. In the first case, we note that u, v are confined to a bounded region when $k \in \mathcal{B}$ and $a \leq \varepsilon$ so that from eq. (3.104) one immediately obtains

$$|F(u,v)| \leq \alpha |v|^\tau. \tag{3.106}$$

In other words, we have

$$|V(k,q;m,a)| \leq \beta a^{-\nu}, \tag{3.107}$$

and since $k^2 + \delta \leq \gamma a^{-2}$ (and $\nu \leq 0$), the desired bound (3.92) follows.

Finally, consider the case $\nu > 0$. Here we must also make use of the assumption that the continuum limit of $V(k,q;m,a)$ exists (property **V2**). In terms of the function F, this condition is equivalent to

$$F(su,sv) \underset{s\to 0}{=} O(s^\omega). \tag{3.108}$$

Note that $\nu > 0$ implies $\omega > 0$ (cf. eq. (3.102)). If we now combine eqs. (3.108) and (3.105), it follows that

$$H_{\mu_1\cdots\mu_\tau}(su,sv) \underset{s\to 0}{=} O(s^\nu). \tag{3.109}$$

Thus, applying Taylor's formula again, we conclude that

$$|F(u,v)| \leq |v|^\tau Q(|u|,|v|) \tag{3.110}$$

where Q is a homogeneous polynomial of degree ν and I have assumed $k \in \mathcal{B}$, $a \leq \varepsilon$ as usual. In other words, we have

$$|V(k,q;m,a)| \leq \alpha Q(|k|,\beta), \tag{3.111}$$

and hence the bound (3.92). This completes the proof of the power counting theorem in the one-loop case.

4. Finite size effects in massive theories

4.1. Basic facts

In numerical simulations of 4-dimensional lattice field theories at zero temperature, one usually employs a $T \times L \times L \times L$ lattice with periodic boundary conditions, where the time-like extent T is ideally much larger than the basic correlation length in the system. The spacial size L typically varies between 10 and 32 lattice spacings and is not easily increased due to memory limitations. This situation is likely to improve in the next few years through the development of more powerful hardware, but the size dependence of the quantities calculated will remain a

problem which has to be carefully analyzed if one wants to have results with a quotable systematic error.

It is instructive to see what the finiteness of the lattice means in physical terms in a concrete case. Consider for example a simulation of lattice QCD on a lattice with $L/a = 20$. To have a reasonably large ultraviolet cutoff $\Lambda_{\mathrm{cut}} = 1/a$, the parameters in the action should be chosen such that the lattice spacing is smaller than (say) 0.1 fm. For the box size we then have $L \leq 2$ fm which is rather small. In particular, a proton on such a lattice is squeezed and its properties (the mass for example) are certainly affected in some way. Another important observation is that the momenta of pions and other particles in this little world are quantized in units of

$$\Delta p = \frac{2\pi}{L} \geq 600\,\mathrm{MeV}. \tag{4.1}$$

This is an uncomfortably high value (recall $m_\pi = 139$ MeV) and it would be a miracle if virtual pion processes were not strongly size dependent under these conditions. Note also that $\Delta p \simeq m_\pi$ would require a lattice size $L \simeq 100\,a$, which is far beyond the sizes which can be accommodated presently.

Finite size effects in theories with only massive particles have often been seen in Monte Carlo simulations. A particularly well studied case is the four-dimensional Ising model, for which Montvay and Weisz [48] have recently produced very accurate data on lattices of variable size L. One quantity they have considered is the mass gap $M(L)$ in the symmetric phase of the model (see fig. 7). Note that the cutoff Λ_{cut} is rather low in this example and values of L, which are relatively large in units of M, are thus affordable. Figure 7 reveals that the finite size effects on the mass gap M are rapidly decreasing for $ML \geq 2$ and are reduced to a fraction of a percent for $ML \geq 5$. I will later provide a theoretical explanation for this behavior and argue that it is typical for the approach to the infinite volume limit in massive theories.

Before I proceed to discuss specific results, I would like to make the following two simple but important remarks.

(a) The energy spectrum in a finite volume is discrete with a level spacing which is often not small. This is certainly the case in the QCD example considered above where the separation between the low-lying n-pion states is essentially determined by the quantum Δp of momentum (and the pion mass). Thus, unless L can be made very much larger than

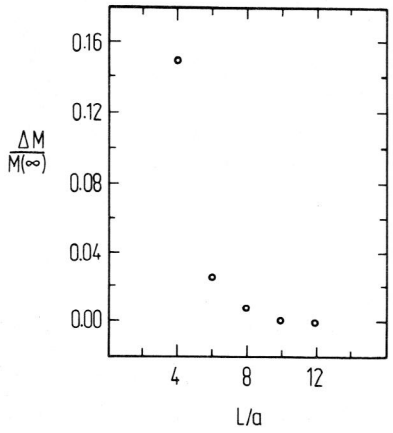


Fig. 7. Plot of the finite size mass shift $\Delta M = M(L) - M(\infty)$ versus the lattice size L in the 4-dimensional Ising model. The nearest neighbor spin coupling is the same for all L and corresponds to $aM(\infty) = 0.4877(5)$.

the Compton wave length of the lightest particle in the theory, it is not to be expected that the poles of correlation functions in the complex energy plane arising from intermediate multi-particle states can mimic a cut singularity very well. However, as I will explain later, the relatively large splitting between successive 2-particle energy levels can also be considered a good property of the finite volume theory, since the L-dependence of these levels provides a handle to compute the strength of the interactions between these particles.

(b) Since finite size effects refer to the scale L, which is usually much larger than the lattice spacing a, one expects that they are not very sensitive to the cutoff Λ_{cut}, but are mainly dependent on

$$z = M(L)L \tag{4.2}$$

(where $M(L)$ denotes the mass gap in the theory) and the dimensionless renormalized couplings of the model. In fact, if the continuum limit

exists in infinite volume, it should normally also be possible to take it at finite values of z. In the following subsections, I will thus frequently present the formulae in a continuum notation, although all of them could also be derived on a lattice with some additional work.

4.2. Volume dependence of stable particle masses in simple models

Consider again the lattice φ^4-theory defined by the action (3.1) and suppose the lattice shape is $T \times L \times L \times L$ with periodic boundary conditions. I would now like to discuss the size dependence of the mass gap $M(L)$, which is defined as usual through the transfer matrix (cf. subsection 2.2). Since the transfer matrix is independent of the time-like extent of the lattice, we may just as well choose $T = \infty$ which is convenient for the calculation that follows.

In this model, single particle states are generated from the vacuum by the fundamental field $\varphi(x)$. Thus, if we define $G_L(p)$ through

$$\langle \varphi(x)\varphi(0)\rangle = \frac{1}{L^3}\sum_{\mathbf{p}} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}p_0}{2\pi} \mathrm{e}^{ipx} G_L(p), \tag{4.3}$$

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}, \quad n_j \in \mathrm{Z}, \quad -\frac{1}{2}L < an_j \leq \frac{1}{2}L, \tag{4.4}$$

the energy $\omega_L(\mathbf{p})$ of the particle with momentum $\mathbf{p}$ in the box is determined by the pole of $G_L(p)$ in the complex p_0-plane closest to the origin, i.e.

$$G_L(p)^{-1} = 0 \quad \text{for } p_0 = \pm i\omega_L(\mathbf{p}). \tag{4.5}$$

The finite volume mass gap is then given by $M(L) = \omega_L(\mathbf{0})$.

In perturbation theory, $G_L(p)$ and $\omega_L(\mathbf{p})$ can be computed straightforwardly. To lowest order the propagator is actually independent of L and given by eq. (3.3). Thus, we have

$$\begin{aligned} \hat{\omega}_L(\mathbf{p}) &\stackrel{\text{def}}{=} \frac{2}{a}\sinh\left(\tfrac{1}{2}a\omega_L(\mathbf{p})\right) \\ &= \sqrt{m^2 + \hat{\mathbf{p}}^2} + O(g). \end{aligned} \tag{4.6}$$

At the next order, there is just one graph (fig. 5a) contributing to the self-energy of φ and the result then is

$$\hat{M}(L) = \sqrt{m^2 + \frac{1}{2}gJ} + O(g^2), \tag{4.7}$$

$$J = \frac{1}{L^3}\sum_{\mathbf{p}} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}p_0}{2\pi} (\hat{p}^2 + m^2)^{-1}. \tag{4.8}$$

This integral depends on L and to see what its behaviour at large L is, we must go through a number of steps.

First we use the Poisson summation formula

$$\frac{1}{L^3}\sum_{\mathbf{p}} f(\mathbf{p}) = \sum_{\mathbf{n}\in Z^3} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^3 p}{(2\pi)^3} f(\mathbf{p}) \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{n}L} \tag{4.9}$$

(which is valid for any smooth periodic function $f(\mathbf{p})$) to show that

$$J = \sum_{\mathbf{n}\in Z^3} \Delta(0, \mathbf{n}L), \tag{4.10}$$

$$\Delta(x) = \int_B \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}px}}{\hat{p}^2 + m^2}. \tag{4.11}$$

Note that $\Delta(x)$ is just the Fourier transform of the (infinite volume) free propagator. In the sum (4.10), the term with $\mathbf{n} = \mathbf{0}$ is independent of L and therefore amounts to a mass renormalization in eq. (4.7).

To evaluate the L-dependent terms, I will make the continuum approximation to be able to bring out the essential structure more clearly (cf. subsection 4.1). The propagator then becomes

$$\begin{aligned} \Delta(x)|_{a=0} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}px}}{p^2 + m^2} \\ &= \frac{1}{8\pi^2|x|} \int \mathrm{d}y\, \mathrm{e}^{-\sqrt{m^2+y^2}|x|}, \end{aligned} \tag{4.12}$$

and for the finite volume mass $M(L)$ one obtains

$$M(L) = m + \frac{g}{16\pi^2} \cdot \frac{3}{mL} \int \mathrm{d}y\, \mathrm{e}^{-\sqrt{m^2+y^2}L} + \cdots, \tag{4.13}$$

where the terms neglected are either of order g^2 or exponentially small compared to the displayed integral as $L \to \infty$. I have also renormalized the mass parameter m such that here it is equal to the physical particle mass in infinite volume and the coupling g can also be identified with (say) the renormalized coupling at zero momentum to this order.

In the Ising limit of the φ^4 theory, eq. (4.13) compares very well with the numerical data of Montvay and Weisz quoted earlier (fig. 7). At the point where they did the Monte Carlo simulation, one has

L/a	$M(L)L$	$(\Delta M/m)_{exp}$	$(\Delta M/m)_{th}$
4	2.248(2)	0.153(2)	0.12(1)
6	3.000(7)	0.025(4)	0.023(3)
8	3.934(6)	0.008(2)	0.006(1)
10	4.883(4)	0.001(2)	0.0015(2)

Table1: Comparison between eq. (4.13) (last column) and results from a numerical simulation of the 4-dimensional Ising model (same data as in fig. 7).

$am = 0.4877(5)$ and $g = 41(5)$. Inserting these values in eq. (4.13) and neglecting the higher order corrections, one obtains the numbers listed in table 1, which show good agreement between theory and experiment. Thus, one may safely conclude that for $mL > 4$, the finite size effects on the single particle mass are below 1% and exponentially decreasing according to eq. (4.13).

The steps which led to the one-loop formula (4.13) can be generalized to any order of perturbation theory and one then finds the following remarkably simple formula for the leading size dependence of the particle mass [44,45]:

$$\Delta M(L) = -\frac{1}{16\pi^2} \cdot \frac{3}{mL} \int \mathrm{d}y \, \mathrm{e}^{-\sqrt{m^2+y^2}L} F(iy) + O(\mathrm{e}^{-\overline{m}L}). \quad (4.14)$$

Here, $\overline{m} \geq \sqrt{3/2}m$, and m denotes the physical particle mass in infinite volume as before. The function F is the analytic continuation of the forward elastic scattering amplitude

$$F(\nu) = \mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 | \mathbf{p}_1, \mathbf{p}_2) \quad (4.15)$$

(cf. eqs(2.52),(2.53)), where

$$\nu = \big(\omega(\mathbf{p}_1)\omega(\mathbf{p}_2) - \mathbf{p}_1 \cdot \mathbf{p}_2\big)/m \quad (4.16)$$

denotes the "crossing variable" and $\omega(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}$ the relativistic expression for the one-particle energy. At large L, the integral in eq. (4.14) receives its dominant contribution from a small interval around $y = 0$ and a saddle point integration then yields $\Delta M \propto L^{-3/2} \exp -mL$. Thus, compared to the simple one-loop result eq. (4.13), no qualitatively new size dependence of the mass gap is found at higher orders of perturbation theory. Essentially, the higher orders can be accounted for by replacing the coupling g in eq. (4.13) by $-F(0)$.

As shown by the derivation of eq. (4.13), the physical origin of the L-dependence of the particle mass in this theory is that the vacuum is polarized by the particle. A virtual particle in the polarization cloud then has some probability to travel around the world and since this process is absent in infinite volume, a small difference in the energy of the polarization cloud (and hence in the observable particle mass) results. It is obvious from this discussion that the effect is quantum field theoretical in nature, in particular, it would not occur in non-relativistic quantum mechanics.

Equation (4.14) generalizes to essentially any theory with only massive particles which can be solved, at low energies, by perturbation theory [44,45]. Depending on the spectrum of light particles and the symmetries in the theory considered, the final formula for the finite size mass shifts may involve several terms of the type (4.14). In particular, in a scalar theory with non-vanishing 3-point coupling λ, a "pole" term

$$-\frac{\lambda^2}{16\pi m} \cdot \frac{3}{mL} e^{-\sqrt{3}mL/2} \tag{4.17}$$

appears, which is more slowly decaying for $L \to \infty$ than the integral (4.14). It is important to appreciate that all these formulae are *universal*, i.e. they do not depend on the details of the interacton Lagrangian, but they only refer to the physical properties (particle spectrum and scattering matrix) of the model. In view of this fact, I believe that these relations are also valid in theories which do not have an infrared stable perturbation expansion. The 2-dimensional non-linear σ-model is such a case and eq. (4.14) has actually been verified numerically [49] (the scattering amplitude is known exactly in these models).

4.3. Models with a spontaneously broken discrete symmetry

A finite size effect of an entirely different kind from the one described above occurs when the system considered has a spontaneously broken discrete symmetry in the infinite volume limit. It is well known that ordinary symmetries cannot be spontaneously broken in a finite volume with periodic boundary conditions, i.e. such symmetries are always represented by well-defined unitary operators which commute with the Hamiltonian, and, in massive theories, the true ground state is usually not degenerate. In particular, it is a singlet under transformations by the elements of the symmetry group G. Nevertheless, spontaneous symmetry breaking leaves a trace at finite L in that every energy eigenstate

is in fact a member of a multiplet of d almost degenerate states with an energy splitting which is much smaller than the typical energy difference between successive multiplets. In general, the degeneracy d is equal to the number of elements of G/H, where H is the subgroup of G which remains unbroken in the infinite volume limit. At $L = \infty$, these finite volume multiplets become exactly degenerate and, through spontaneous choice of one of the ground states, the degeneracy is effectively removed from the spectrum, because it is impossible to reach the other states by local processes. If one is primarily interested in the properties of the model as $L \to \infty$, such as the particle mass spectrum, one should therefore be careful to not mix up one-particle states with the states that are almost degenerate with the ground state and which ultimately decouple. For numerical simulations, this is potentially a problem, since the gap between these lowest states can be very small and hence introduces a correlation length in the system much larger than the one associated with the "physical" particles.

To obtain a better understanding of the effect described above, consider again the φ^4-theory, but now we choose $m^2 < 0$ so that the model is in the phase where the reflection symmetry $\varphi \to -\varphi$ is spontaneously broken. For $L = \infty$, the field φ then assumes a vacuum expectation value

$$\langle\varphi\rangle = \sqrt{-6m^2/g} + O(\sqrt{g}), \tag{4.18}$$

and the physical particle mass is $m' = \sqrt{-2m^2} + O(g)$. The finite volume ground state, on the other hand, is invariant under $\varphi \to -\varphi$, and there is another state with a small energy ΔE above the ground state, which is odd under this symmetry. If we denote these states by $|+\rangle$ and $|-\rangle$ respectively, we have

$$\langle +|\varphi|+\rangle = \langle -|\varphi|-\rangle = 0, \tag{4.19}$$

$$|\langle +|\varphi|-\rangle| \underset{L\to\infty}{\sim} \langle\varphi\rangle. \tag{4.20}$$

Accordingly, the two-point correlation function is, for large x_0, given by

$$\langle\varphi(x)\varphi(0)\rangle = |\langle +|\varphi|-\rangle|^2 \, \mathrm{e}^{-\Delta E x_0}. \tag{4.21}$$

On the basis of this relation, it is possible to compute ΔE by numerical simulation and also the matrix element $|\langle +|\varphi|-\rangle|$, which can be considered a finite volume expression for the vacuum expectation value of φ. Such a calculation has just been completed by Jansen et al. [53,54] and

L/a	$\Delta E/m'$	$\tilde{\sigma}/m'^3$
6	0.327(2)	0.0860(9)
7	0.207(1)	0.0762(8)
8	0.1176(7)	0.0694(6)
9	0.0571(4)	0.0652(6)
10	0.0230(2)	0.0626(6)

Table 2: Values of the finite volume energy gap ΔE and the "surface tension" (4.26) from a numerical simulation of the 4-dimensional Ising model [53,54] at a point where the physical particle mass m' is about 0.4 in lattice units.

in table 2 I show some of their data (2'nd column). The simulation was done in the Ising limit of the φ^4-theory in four dimensions at a value of the nearest neighbor spin coupling which corresponds to a physical particle mass m' given by $am' = 0.392(1)$. I emphasize that it is not easy to obtain these numbers, because ΔE is small in lattice units and one must therefore be able to follow the exponential decay (4.21) of the two-point function over distances of about 10 – 40 lattice spacings.

From table 2 one sees that ΔE is a very rapidly decreasing function of L and when $m'L \simeq 4$, it is at most a few percent of the particle mass m'. Qualitatively, this behaviour can be understood from the following simple argument [50-52].

A typical field configuration in the statistical ensemble of the φ^4 theory on a lattice with $T \gg L$ looks schematically as in fig. 8. There are relatively large intervals in time, where

$$\bar{\varphi}(x_0) = \frac{a^3}{L^3} \sum_{\mathbf{x}} \varphi(x) \tag{4.22}$$

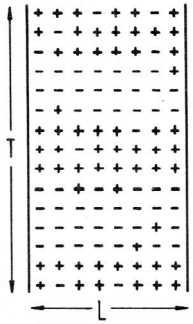
is close to either $+\langle\varphi\rangle$ or $-\langle\varphi\rangle$. Between these regions, $\bar{\varphi}(x_0)$ changes sign, i.e. there is a Bloch wall. For large L, the free energy F_{BW} associated to such a Bloch wall is proportional to the volume,

$$F_{BW} = \sigma L^3, \tag{4.23}$$

and, in the continuum approximation, it is easy to show that

$$\sigma = 2m'^3/g + O(1). \tag{4.24}$$

Furthermore, the width of a Bloch wall is determined by the scale m' and it thus follows, that Bloch walls are dilute at large L.



← Bloch wall

Fig. 8. Typical field configuration in the φ^4-theory in the broken symmetry phase ("+" and "−" indicate a positive and negative value of $\varphi(x)$ respectively).

It is obvious from the above that the Bloch walls destroy the long range order of the system and, by making a dilute gas Ansatz, the estimate

$$\Delta E/m' = A(m'L)^{\frac{1}{2}}\mathrm{e}^{-\sigma L^3} \tag{4.25}$$

is obtained for the associated correlation length $(\Delta E)^{-1}$ [51,52].

The exponential factor in eq. (4.25) explains why ΔE is so rapidly vanishing as L increases. If we define

$$\tilde{\sigma} = -\frac{1}{L^3}\ln(\Delta E/m'), \tag{4.26}$$

eq. (4.25) suggests that $\lim_{L\to\infty}\tilde{\sigma} = \sigma$. As shown by table 2, the numerical results of Jansen et al. for $\tilde{\sigma}$ are indeed slowly varying as a function of L and apparently converge with increasing L. Furthermore, the value of $\tilde{\sigma}/m'^3$ on the largest lattice happens to be rather close to the estimate $\sigma/m'^3 = 0.065(2)$, which one obtains from the tree level formula (4.24) by inserting the measured value $g = 31(1)$ for the renormalized

coupling [1]. Thus, little doubt remains that the theoretical explanation of the energy gap ΔE is correct and this then allows one to decide how large the lattice volume must be to make sure that this kind of finite size effect has a negligible influence on e.g. the (physical) particle spectrum and the vacuum expectation value of the field.

4.4. The case of (pure) non-Abelian gauge theories

The size dependence of glueball masses in SU(2) and SU(3) gauge theories (as defined in subsection 2.1), has been a debated topic for many years now and it is not my intention here to review all the important contributions which have been made. In particular, the analytic glueball calculations are complicated and to explain them in any detail would require an additional set of lectures. Thus, I shall only summarize some basic facts and illustrate them by the most recent numerical results.

From the definition of the transfer matrix T and the Hilbert space $\mathcal{H}$ of physical states discussed in subsection 2.1, it is obvious that the proper lattice rotations R are symmetries of the system which can be represented by unitary operators commuting with T. The group SO(3, Z) of all these rotations has 24 elements and 5 irreducible representations as listed in table 3. Further symmetries are parity and the charge conjugation operation $V(\mathbf{x}, k) \to V(\mathbf{x}, k)^*$, which is equivalent to a gauge transformation in the SU(2) case. Since these transformations commute with the cubic rotations, the glueball states at zero momentum can be labelled by quantum numbers Γ^{PC}, where Γ denotes an irreducible representation of SO(3, Z) and $P = \pm$, $C = \pm$ are the parity and charge conjugation eigenvalues. Note that this classification applies independently of the lattice size L.

In addition to the symmetries listed so far, there exist further transformations, first described by t' Hooft [58], which commute with the transfer matrix and which are associated with our choice of periodic boundary conditions for the gauge field, i.e. they do not survive in the infinite volume limit. They are, however, important for the understanding of the energy spectrum at finite L and I thus discuss them here in some detail.

---

[1] Since these lectures have been delivered, the O(1) correction in eq. (4.24) and also the constant A in the semi-classical formula (4.25) have been worked out by Münster [52]. A more careful analysis of the numerical data is hence possible and an even more impressive matching between theory and "experiment" is then observed [54].

name	dimension	tensor model
A_1	1	trivial representation
A_2	1	t_{klj}, $t_{klj} \propto \lvert\epsilon_{klj}\rvert$
E	2	t_{kl}, $t_{kl} = 0$ for $k \neq l$
T_1	3	t_k
T_2	3	t_{kl}, $t_{kl} = 0$ for $k = l$

Table 3: Irreducible representations of the cubic group $\mathrm{SO}(3,\mathrm{Z})$. $R \in \mathrm{SO}(3,\mathrm{Z})$ acts on tensors $t_{k\ldots j}$ in the usual way and the tensors t_{kl} occurring in the models for E and T_2 are assumed to be trace-less and symmetric.

Suppose $z = (z_1, z_2, z_3)$ is a triplet of elements of Z_N,

$$z_k = \mathrm{e}^{i2\pi\nu_k/N}, \quad \nu_k \in \{0, 1, \ldots, N-1\}, \tag{4.27}$$

and define

$$\Lambda_z(\mathbf{x}) = \exp(\frac{2\pi}{L}\nu \cdot \mathbf{x}\, W), \tag{4.28}$$

where W is the diagonal $N \times N$ matrix with

$$W_{\alpha\alpha} = \frac{i}{N}(1 - N\delta_{\alpha N}). \tag{4.29}$$

Since W is trace-less and anti-hermitian, $\Lambda_z(\mathbf{x})$ is in $\mathrm{SU}(N)$. Furthermore, we have

$$\Lambda_z(\mathbf{x} + L\hat{k}) = z_k \Lambda_z(\mathbf{x}). \tag{4.30}$$

It follows that the transformation

$$V(\mathbf{x}, k) \to C_z V(\mathbf{x}, k) = \Lambda_z(\mathbf{x}) V(\mathbf{x}, k) \Lambda_z(\mathbf{x} + \hat{k})^{-1} \tag{4.31}$$

maps a periodic gauge field onto a periodic one and

$$(U_z\psi)[V] = \psi[C_z^{-1}V] \tag{4.32}$$

thus defines a unitary operator U_z in $\mathcal{H}$ (it is not difficult to check that $U_z\psi$ is gauge invariant when ψ has this property).

For any choice of $z \in \mathrm{Z}_N^3$, the associated operator U_z commutes with the transfer matrix T and we have thus established the group Z_N^3 as a further symmetry of the system. At first sight one might think that the transformation (4.31) is just a gauge transformation, but this is actually

not the case, because $\Lambda_z(\mathbf{x})$ is not periodic. This implies, for example, that Wilson loops winding around the world change by a phase z_k under the action of C_z.

The possible eigenvalues of the operators U_z are characters of Z_N^3 and are thus given by

$$U_z\psi = (z_1)^{e_1}(z_2)^{e_2}(z_3)^{e_3}\psi, \quad e_k \in \mathrm{Z}\,(\mathrm{mod}\ N). \tag{4.33}$$

There are N^3 different choices for the quantum numbers e_k (which 't Hooft called "electric fluxes") and the Hilbert space $\mathcal{H}$ accordingly divides into N^3 sectors, which will later be referred to as the "electric flux" sectors.

The ground state $|0\rangle$ of the theory can be shown to have quantum numbers A_1^{++} and $\mathbf{e} = \mathbf{0}$ (recall that for $L < \infty$, the spectrum of the Hamilton operator H is discrete and the ground state is unique). As discussed in section 2, glueball states are obtained by applying linear combinations of local fields to the vacuum state and since local fields commute with the transformations U_z, it follows that these states are also contained in the zero electric flux sector. States with $|\mathbf{e}| = 1$ can be generated from the vacuum by acting with a Wilson loop which winds around the torus once in a space-like direction. This suggests that the physical interpretation of these states is that they describe a flux tube going around the world plus a number of glueballs or perhaps other local excitations. In particular, the ground state energy ΔE in the sectors with $|\mathbf{e}| = 1$ (which have an identical energy spectrum because of rotational invariance), is considered to be the energy of a flux tube of length L.

The above interpretation has actually been confirmed for strong coupling g_0 where Münster [59] has been able to prove that

$$\Delta E \underset{L\to\infty}{=} \sigma L + O(1), \tag{4.34}$$

where σ denotes the string tension which one defines through the static quark anti-quark potential. Thus, most lattice gauge theorists take eq. (4.34) for granted, even at smaller values of g_0, where Münster's proof does not apply.

Incidentally, it is interesting to note that if we assume the string model to give an accurate description of the flux tube at large L, we would infer that [60,61]

$$\Delta E \underset{L\to\infty}{=} \sigma L + \kappa - \frac{\pi c}{6L} + O(1/L^2), \tag{4.35}$$

where κ is a constant and c is the central charge of the string model. In the simplest possible case, we just have two bosonic degrees of freedom associated with the transverse fluctuations of the flux tube and hence $c = 2$. In the string model, κ is not universal, but in the gauge theory, ΔE is an energy difference and the dimensionless ratio $\kappa/\sqrt{\sigma}$ should therefore approach a well-defined number in the continuum limit.

I now proceed to discuss some of the results on the energy spectrum which have been obtained by perturbation theory and semi-classical methods. First, it is possible to show that all eigenvalues of H can be expanded in a power series of $\bar{g}^{2/3}$, where $\bar{g}$ is a renormalized "running" gauge coupling at scale L. For example, for the mass $M(A_1^{++})$ of the lightest glueball with quantum numbers A_1^{++} (and electric flux $\mathbf{e} = \mathbf{0}$), we have the asymptotic expansion

$$M(A_1^{++}) \sim \frac{1}{L} \sum_{k=1}^{\infty} \varepsilon_k \, [\bar{g}(L)]^{2k/3}, \tag{4.36}$$

where the coefficients ε_k are numerically known, for the SU(2) theory, up to one-loop order (i.e. for $k \leq 4$) [62,63]. It is perhaps a little surprising that broken powers of $\bar{g}$ appear in this expansion, but this is due to the fact that some of the modes of the system are not harmonic to lowest order of perturbation theory. More precisely, when one expands about the classical vacuum configuration $V(\mathbf{x}, k) = 1$, the potential energy of the constant modes c_k is found to be proportional to

$$-\sum_{k,l} \mathrm{tr}\,\{[c_k, c_l]\,[c_k, c_l]\}\,, \tag{4.37}$$

i.e. there is no term proportional to $\mathrm{tr}\,\{c_k^2\}$. Thus, the situation is exactly as in a purely anharmonic oscillator, where the level splitting is proportional to $g^{2/3}$, as one may easily show by a rescaling of variables.

For the proper use of eq. (4.36), it is important to note that the expansion is actually only valid when L is small compared to the dynamically generated length scale Λ^{-1} in this theory. The reason for this is that $\bar{g}(L)$ is, according to asymptotic freedom, approaching zero for $L \to 0$,

$$\bar{g}^2 \underset{L \to 0}{\sim} -[2\beta_1 \ln \Lambda L]^{-1}\,, \quad \beta_1 = \frac{11}{48\pi^2} N, \tag{4.38}$$

while for $L \sim \Lambda^{-1}$, the coupling is presumably of order 1 and the perturbation series (4.36) is then not applicable. Thus, as was to be expected,

perturbation theory in this model tells us nothing about the approach to the infinite volume limit and, in this respect, the situation is therefore exactly opposite to what we found in the cases studied previously.

Still, it is interesting to see what the glueball spectrum at small L is, because this provides an opportunity to compare with numerical simulations and, furthermore, it is also instructive to see how the individual levels evolve as the volume is increased. It has become customary to express the glueball mass ratios as a function of

$$z = M(A_1^{++})L, \tag{4.39}$$

which is a measure of the box size in units of the correlation length in the A_1^{++} sector. Note that in the continuum limit, the mass ratios are universal functions of z alone, in particular, the gauge coupling $\bar{g}$ drops out in these relations.

From the apparent convergence of the perturbation expansion in the SU(2) case [63], one would conclude that it applies up to about $z = 1.5$, but later on I will show that already at $z = 1$ there is an interesting non-perturbative phenomenon happening, which invalidates the perturbative analysis. Thus, the range where perturbation theory applies is

$$0 \leq z \leq 1. \tag{4.40}$$

For these values of z, the glueball spectrum in the SU(2) theory is as follows [63] (similar results are obtained for SU(3) [71], but the calculations are less complete in this case). Below $2M(A_1^+)$ there are 6 glueballs with quantum numbers and masses (at z=1) as listed in table 4 [2]. For $z \leq 1$, the mass ratios $M/M(A_1^+)$ of these particles are practically independent of z. Negative parity energy eigenstates also exist, but only above the "two-particle" threshold $2M(A_1^+)$.

An interesting aspect of table 4 is that the β and γ glueballs, which transform non-trivially under the cubic group, are actually the lightest glueballs at $z = 1$, although they are not very much lighter than the α-particle. Another observation is that the β, γ and β', γ' states are almost degenerate. If such a degeneracy would occur at $L = \infty$, the interpretation would simply be that the β and γ glueballs are just different spin states of a tensor glueball with spin 2 (from table 3 one quickly sees that

[2] In the SU(2) theory, charge conjugation is a trivial operation (all states have $C = +$) and C is hence not indicated in this case.

name	Γ^P	$M/M(A_1^+)$
α	A_1^+	1.00
β	E^+	0.86
γ	T_2^+	0.86
α'	A_1^+	1.65
β'	E^+	1.56
γ'	T_2^+	1.57

Table 4: Glueball spectrum in the SU(2) theory at $z = 1$. All other states have a mass $M \geq 2M(A_1^+)$.

the $J = 2$ representation of SO(3) decomposes into $E \oplus T_2$, when the group is reduced to SO(3, Z)). At $z = 1$, however, no such interpretation is possible and the degeneracy must be considered accidental.

So far I have been discussing states with zero electric flux. It turns out, however, that the energy spectrum in the sectors with $\mathbf{e} \neq \mathbf{0}$ is actually exactly the same, to all orders of perturbation theory, as in the zero flux sector [62]. In particular, the flux tube energy ΔE vanishes to all orders of $\bar{g}$. The origin of this degeneracy is that for small $\bar{g}$, the ground state wave function $\psi_0[V]$ is essentially supported in the neighborhood of the classical vacuum $V_0(\mathbf{x}, k) = 1$ and the configurations $C_z V_0$. Thus, ψ_0 consists of N^3 pieces which do practically not overlap and whose relative phases can therefore be chosen arbitrarily without altering the energy of the state. In this way one obtains the ground states in the sectors with an arbitrary electric flux $\mathbf{e}$ and these are therefore degenerate. Evidently, the situation is similar to the one encountered in the double well anharmonic oscillator, where the even and odd ground states are also degenerate to all orders of perturbation theory.

At a non-perturbative level, tunneling between the classical vacua $C_z V_0$ sets in and the degeneracy of the electric flux sectors is lifted. The calculation of the associated tunneling amplitude is, for various technical reasons, extremely difficult and it is only after developing adequate methods in simpler models that van Baal and Koller have been able to solve this problem recently [64-69]. Their result for the flux tube energy ΔE in the SU(2) theory reads

$$\Delta E/M(A_1^+) = 0.00767 \cdot z^{3/2} \exp\left(-42.6169 \cdot z^{-3/2} + 34.2001 \cdot z^{-1/2}\right) \times \left(1 + O(z^{1/4})\right), \tag{4.41}$$

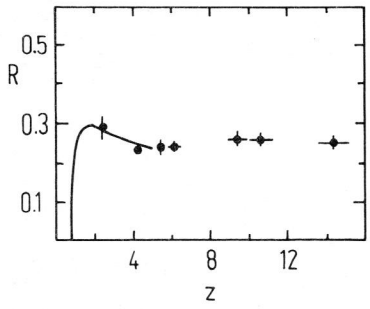


Fig. 9. Plot of the ratio $R = \sqrt{\Delta E L}/z$ for the SU(2) theory. The full line is the result of the analytic calculation by van Baal and Koller, while the points are obtained by numerical simulation [75,77,78].

which gives $\Delta E/M(A_1^+) = 0.1$ at $z = 1.34$. When z is greater than this value, ΔE is very rapidly rising and at (say) $z \geq 1.5$, the semi-classical formula (4.41) can no longer be trusted.

In view of this singular behaviour, van Baal and Koller thought of a better although not equally rigorous method to compute ΔE for which they have reason to believe that it applies up to $z = 5$. The basic idea is to first identify the degrees of freedom which are most relevant for the tunneling process. These have been known by the name of "torons" for some time: they are just the constant Abelian gauge fields. Next, one derives an effective Hamiltonian H′ for the torons by "integrating out" the other degrees of freedom systematically in perturbation theory. Finally, H′ is diagonalized exactly in a Hilbert space of wave functions which obey certain boundary conditions as appropriate for the quantum number sector one wants to consider (in the trivial but analogous case of an anharmonic oscillator with a reflection symmetric potential $V(x)$, the diagonalization would be performed in the spaces of wave functions $\psi(x)$, $x \geq 0$, with Dirichlet respectively Neumann boundary conditions at $x = 0$).

A plot of ΔE as calculated by the method outlined above, together

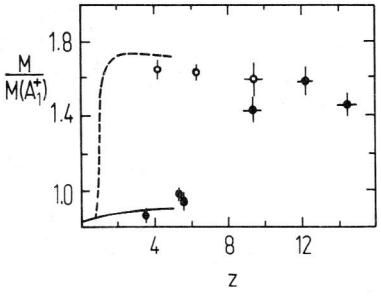


Fig. 10. Glueball mass ratios in the SU(2) theory. The solid line and the full circles refer to the β particle, the dashed line and the open circles to the γ particle. The lines are from the analytical study by van Baal and Koller (with an amendment by Vohwinkel [70]) and the numerical data are taken from refs.[75,77,78] .

with some recent Monte Carlo points, is shown in fig. 9 [3]. The agreement between theory and "experiment" is impressive, all the more so, since no parameters are adjusted in these calculations. From the figure it also appears that after the tunneling transition at $z \simeq 1.4$, there is only little size dependence so that the infinite volume estimate $\sqrt{\sigma}/M(A_1^+) = 0.25(3)$ can be extracted with confidence. In view of this situation, it is tempting to conclude that infinite volume behaviour is generally established for (say) $z \geq 4$ as in the scalar theories we have considered earlier.

Unfortunately, this does not seem to be the case, because if we now consider the size dependence of glueball mass ratios (fig. 10), it appears that the ratio of the β to the α mass is rising from around 1 at $z = 5$ to about 1.5 at $z = 10$. While this result is based on only few Monte Carlo

[3] The literature on numerical glueball mass calculations in pure SU(2) and SU(3) gauge theories is extensive [72-85]. In figs.9 and 10, I only show a small fraction of the available data, for the purpose of illustration. For a review of the status of glueball mass calculations, see [86]

points, it seems unavoidable that something happens above $z = 5$, since the E^+ and T_2^+ states must pair at large L to form a tensor glueball with angular momentum $J = 2$, as explained above. However, one should not be jumping to conclusions because a number of uncertainties still exists. In particular, due to memory limitations, simulations at large z are done on lattices, where the cutoff $\Lambda = 1/a$ is almost as low as $M(A_1^+)$, so that it would be rather strange if the mass ratios would be unaffected by short distance lattice effects. Furthermore, one should make sure that the excited states β' and γ' are not confused with β and γ, especially in the intermediate z range, and finally the exponential approach to the infinite volume limit, as described in subsection 4.2, should be observed.

It is obvious that the study of these questions requires precision calculations of glueball masses on large lattices of variable size. Using conventional numerical techniques, such calculations would be very expensive so that to make significant progress in this field, one probably has to develop more effective updating algorithms, and also the operators used as interpolating fields need to be improved to yield better signal to noise ratios. The history of glueball mass calculations however shows that the pure non-Abelian gauge theories are more complicated (and also more interesting) than expected and it is therefore well worth to study them in even greater details than was possible so far.

4.5. Two-particle states in finite volume

In a free field theory of massive particles, the possible energy values W of two-particle states in finite volume are

$$W = \sum_{i=1}^{2} \sqrt{m_i^2 + \mathbf{p}_i^2}, \tag{4.42}$$

where m_i and $\mathbf{p}_i = (2\pi/L)\mathbf{n}_i$, $\mathbf{n}_i \in Z^3$, are the masses and momenta of the two particles. As emphasized in subsection 4.1, the quantum of momentum $\Delta p = 2\pi/L$ is often not so small. In such a situation the splitting of the levels (4.42) is sizeable and it is therefore possible, in a numerical simulation, to determine these energy values from a calculation of suitable four-point correlation functions. One may expect (and we shall show) that this feature persists in the presence of interactions. Thus, it is sensible to study the volume dependence of the individual two-particle energy levels and we shall see that this results in an interesting possibility to determine the scattering lenghts associated to the

elastic scattering processes in the infinite volume theory. What follows is essentially a summary of [46] and the reader is referred to this paper for further details and proofs.

To find out how the two-particle energy levels (4.42) are affected by interactions, I shall again use perturbation theory. It is possible to carry out these calculations in the framework of quantum field theory, using Feynman diagrams, but it is perhaps more instructive (and certainly less complicated) to study the question in non-relativistic quantum mechanics. Thus, suppose we have two identical, non-relativistic particles of mass m and spin 0 enclosed in a periodic box of size L. Such a two-particle state can be described by a scalar wave function $\psi(\mathbf{x},\mathbf{y})$ which satisfies

$$\psi(\mathbf{x},\mathbf{y}) = \psi(\mathbf{y},\mathbf{x}), \tag{4.43}$$

and which is periodic with period L in all components of the arguments $\mathbf{x}$ and $\mathbf{y}$. The Hamilton operator H of the system is assumed to be of the form

$$\mathrm{H} = \mathrm{H}_0 + \mathrm{V}, \tag{4.44}$$

where the action of H_0 and V on wave functions ψ is given by

$$\mathrm{H}_0\psi(\mathbf{x},\mathbf{y}) = -\frac{1}{2m}(\Delta_x + \Delta_y)\psi(\mathbf{x},\mathbf{y}), \tag{4.45}$$

$$\mathrm{V}\psi(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{n}\in Z^3} \mathcal{V}(\mathbf{x}-\mathbf{y}+\mathbf{n}L)\psi(\mathbf{x},\mathbf{y}). \tag{4.46}$$

Here, Δ_x, Δ_y denote the Laplace operators with respect to $\mathbf{x}$ and $\mathbf{y}$, and the (real) function $\mathcal{V}(\mathbf{z})$ is assumed to be short ranged.

Since the Hamiltonian is translationally invariant, the total momentum operator $\mathbf{P}$ commutes with H. In what follows, we only consider the eigenstates of H with zero total momentum (no new interesting aspects show up when $\mathbf{P} \neq \mathbf{0}$). The corresponding eigenfunctions of the free Hamilton operator H_0 are the symmetrized planes waves

$$\psi_{\mathbf{p}}(\mathbf{x},\mathbf{y}) = \mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}, \tag{4.47}$$

which will be written as $|\mathbf{p}\rangle$ in Dirac's notation. Thus, we have

$$\mathrm{H}_0|\mathbf{p}\rangle = 2\epsilon(\mathbf{p})|\mathbf{p}\rangle, \tag{4.48}$$

$$\epsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}, \tag{4.49}$$

and the normalization may be chosen such that

$$\langle \mathbf{p}'|\mathbf{p}\rangle = \delta_{\mathbf{p}'\,\mathbf{p}} + \delta_{\mathbf{p}'\,-\mathbf{p}}. \tag{4.50}$$

In this basis, the matrix elements of the interaction operator V are given by

$$\langle \mathbf{p}'|\mathrm{V}|\mathbf{p}\rangle = \frac{1}{L^3}\tilde{\mathcal{V}}(\mathbf{p}', \mathbf{p}), \tag{4.51}$$

$$\tilde{\mathcal{V}}(\mathbf{p}', \mathbf{p}) = \int_{-\infty}^{\infty} \mathrm{d}^3 z \left\{ \mathrm{e}^{-i(\mathbf{p}'-\mathbf{p})\mathbf{z}} + \mathrm{e}^{-i(\mathbf{p}'+\mathbf{p})\mathbf{z}} \right\} \mathcal{V}(\mathbf{z}). \tag{4.52}$$

In the absence of interactions, the lowest two-particle energy value is obtained for zero relative momentum ($\mathbf{p} = \mathbf{0}$). Since this is a non-degenerate eigenvalue of H_0, perturbation theory is straightforward and, to second order in the interaction V, the corresponding exact eigenvalue E of the full Hamiltonian is found to be

$$E = \frac{1}{2L^3}\left\{ \tilde{\mathcal{V}}(\mathbf{0}, \mathbf{0}) - \frac{1}{2L^3} \sum_{\mathbf{k}\neq\mathbf{0}} \tilde{\mathcal{V}}(\mathbf{0}, \mathbf{k}) \frac{m}{\mathbf{k}^2} \tilde{\mathcal{V}}(\mathbf{k}, \mathbf{0}) + \cdots \right\}. \tag{4.53}$$

This formula already tells us that the interaction implies a deviation from the free particle energy $E = 0$ by an amount of order L^{-3}, which is small compared to the next-to-lowest two-particle energy, at least when L is sufficiently large. For $L \to \infty$, the sum in eq. (4.53) may be replaced by an integral and we then have

$$E = \frac{1}{2L^3}\left\{ \tilde{\mathcal{V}}(\mathbf{0}, \mathbf{0}) - \frac{1}{2} \int \mathrm{d}^3 k \; \tilde{\mathcal{V}}(\mathbf{0}, \mathbf{k}) \frac{m}{\mathbf{k}^2} \tilde{\mathcal{V}}(\mathbf{k}, \mathbf{0}) + \cdots \right\}. \tag{4.54}$$

up to terms of order L^{-4}.

The curly bracket in eq. (4.54) is independent of L and is in fact related to the scattering matrix in infinite volume for particles interacting through the potential $\mathcal{V}(\mathbf{x} - \mathbf{y})$. Indeed, the terms shown are just the first two contributions to the Born series for the scattering matrix at zero momentum and the result thus is

$$E = -\frac{4\pi a_0}{mL^3} + O(\mathcal{V}^2 L^{-4}) + O(\mathcal{V}^3), \tag{4.55}$$

where a_0 denotes the scattering length for S-wave elastic particle scattering.

L/a	$(\Delta W/2m)_{exp}$	$(\Delta W/2m)_{th}$
4	0.364(8)	0.468
6	0.124(7)	0.117
8	0.046(8)	0.045
10	0.022(5)	0.022

Table 5: Comparision between eq. (4.57) (last column) and data obtained by a numerical simulation of the four dimensional Ising model [48].

The asymptotic formula (4.55) can be improved and generalized in several respects. First, the analysis can be carried on to any order of perturbation theory. Second, the corrections of order L^{-4} (and higher) can be determined by keeping the subleading terms when taking the large L limit of the momentum sums involved in the perturbation expansion. For example, for the sum appearing above, we have

$$\frac{1}{L^3}\sum_{\mathbf{k}\neq 0}\tilde{\mathcal{V}}(\mathbf{0},\mathbf{k})\frac{m}{\mathbf{k}^2}\tilde{\mathcal{V}}(\mathbf{k},\mathbf{0}) = \int \mathrm{d}^3k\ \tilde{\mathcal{V}}(\mathbf{0},\mathbf{k})\frac{m}{\mathbf{k}^2}\tilde{\mathcal{V}}(\mathbf{k},\mathbf{0}) + \frac{c}{L}m[\tilde{\mathcal{V}}(\mathbf{0},\mathbf{0})]^2 + O(1/L^2), \tag{4.56}$$

where $c = -0.225\,784\,96$ is a geometric constant, similiar to Madelung's constant, which is related to the three dimensional momentum lattice. Finally, the whole derivation can be repeated for relativistic quantum field theories. After all these steps are taken into account, the result for relativistic identical bosons reads [46,47]

$$W = 2m - \frac{4\pi a_0}{mL^3}\left\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right\} + O(L^{-6}), \tag{4.57}$$

$$c_1 = -2.837\,297, \tag{4.58}$$

$$c_2 = \ 6.375\,183, \tag{4.59}$$

a_0 being the scattering length as before. Note that all explicit reference to the potential has disappeared from this formula, i.e. its form is universal similarly to what we have found in the case of the single particle finite volume mass shift.

In the numerical study of the four-dimensional Ising model by Montvay and Weisz quoted earlier (cp. subsection 4.2), not only the size

dependence of the particle mass was determined, but it also turned out that a calculation of the lowest two-particle energy W is feasible by a straightforward analysis of the exponential falling off of a suitable four-point function of the spin field at large times. At the point where the data were taken, the particle mass m and the scattering length a_0 are approximately equal to 0.4877 and -0.68 in lattice units (a_0 is computed from the measured values of m and the renormalized coupling at zero momentum, using renormalized perturbation theory up to two loops). Inserting these values into eq. (4.57), one obtains column 3 of table 5, where I have defined $\Delta W = W(L) - 2m$. The agreement with the "experimental" values is very good for $L/a \geq 6$. Note that by table 1, the finite size mass shift $\Delta M/m$ is already rather small and exponentially decreasing for these values of L.

In the above example, it is obvious that the scattering length a_0 could have been determined rather well from the numerical data alone by fitting the two-particle energy values W for $L/a \geq 6$ with the asymptotic formula (4.57), a_0 being the fit parameter (the mass m is assumed to be known from the exponential decay of the spin two-point function). Thus it has been demonstrated that the observation of finite size effects can be used to calculate certain scattering matrix elements, and, of course, one hopes that the method will also be useful in other more complicated theories where the scattering matrix is not known beforehand from analytic studies.

Finally, I would like to remark that eq. (4.57) can be generalized easily to the higher two-particle states (those where the particles have relative momentum $\mathbf{p} \neq \mathbf{0}$) and it is also possible to treat spinning particles in much the same way. For an application to the $\pi\pi$- and πN-system see [46] (finite size effects in QCD, in various kinematical situations, have also been studied recently by Leutwyler and Gasser [55,56], using chiral perturbation theory).

4.6. Resonances

In QCD, the Higgs model and in many other theories of physical interest, the occurence of unstable particles (resonances) is a common phenomenon. Their properties are often well-known experimentally and it is therefore important to compute them when a solution of the theory is attempted by (say) a numerical simulation of the corresponding lattice model. A conceptual difficulty then is that a resonance does not in any simple way correspond to a well-defined eigenvalue of the Hamiltonian. In particular, without further insight, one should not expect that

e.g. the ρ-meson mass in (unquenched) QCD can be determined simply by computing an appropriate two-point correlation function on a finite lattice and looking for its exponential decay at large times as if the ρ was a stable particle.

In this last section, I would like to show that a resonance leads to some peculiar finite size effects on the energy spectrum around the resonance energy. It could well be that these are observable in numerical simulations when the resonance is sufficiently narrow and if simulations on variable sized lattices can be afforded. In this case, the mass and perhaps the width of the resonance would be calculable. The discussion that follows is mainly qualitative; further details and formulae will be given in [57].

To get a first impression on what the finite volume signatures of a resonance can be, it is useful to study a very simple quantum-mechanical model of two (identical) scalar particles moving on a line. Thus, the system considered is as described at the beginning of subsection 4.5, the only difference being that now the particle coordinates x and y are one-dimensional. At zero total momentum, the wave function of the particles depends only on the relative coordinate $z = x - y$ and the stationary Schrödinger equation reduces to

$$\{-\frac{1}{m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mathcal{V}(z)\}\psi(z,0) = E\psi(z,0), \qquad |z| \le \frac{1}{2}L, \tag{4.60}$$

where I have assumed, for simplicity, that $\mathcal{V}$ has finite range $R < L/2$ so that the terms with $n \neq 0$ do not contribute in eq. (4.46). For any energy value E and in infinite volume, eq. (4.60) has a unique solution $\psi(z,0) = \phi_E(z)$ which respects the Bose symmetry (4.43), i.e. which is symmetric under $z \to -z$. In the range $|z| > R$, the form of this solution is (up to normalization)

$$\phi_E(z) = \mathrm{e}^{-\mathrm{i}p|z|} + \mathrm{e}^{2\mathrm{i}\delta}\mathrm{e}^{\mathrm{i}p|z|}, \tag{4.61}$$

where $p = \sqrt{mE}$ is the meson momentum and $\delta(p)$ the scattering phase shift.

In finite volume, $\phi_E(z)$ is still a solution of the Schrödinger equation, but now not all (positive) energy values are allowed because the boundary conditions

$$\phi_E(-L/2) = \phi_E(L/2), \qquad \phi'_E(-L/2) = \phi'_E(L/2), \tag{4.62}$$

must be satisfied. In view of eq. (4.61), this amounts to

$$e^{2i\delta(p)}e^{ipL} = 1, \tag{4.63}$$

which is a non-linear equation for the momentum p. The possible finite volume energy values E are thus given by $E = p^2/m$, where p is any solution of eq. (4.63).

Suppose now that the potential $\mathcal{V}$ is such that for some momentum $p = p_R$ the scattering of the two particles on the line is resonant. Around the resonance, the scattering phase shift then assumes the form

$$\delta(p) = \delta_B(p) + \delta_R(p), \tag{4.64}$$

where δ_B is a smoothly varying function, the "background" phase shift, and δ_R the resonance contribution, viz.

$$\delta_R(p) = \frac{1}{2i} \ln \frac{E_R - E + i\Gamma/2}{E_R - E - i\Gamma/2}. \tag{4.65}$$

Here, E_R denotes the resonance energy, $E_R = p_R^2/m$, and Γ is the width of the resonance which I assume to be much smaller than E_R. $\delta_R(p)$ is close to zero for energies well below the resonance and then rises very rapidly in the resonance region towards the high energy value $\delta_R = \pi$.

For a phase shift as specified above, the pattern of the finite volume energy levels around the resonance energy E_R can be determined graphically as follows (see fig. 11). As explained above, we only have to solve eq. (4.63), which I now rewrite in the form

$$pL = 2\pi n - 2\delta(p), \tag{4.66}$$

where n is any integer. For p close to p_R, we may assume that the background phase shift $\delta_B(p)$ is independent of p to a first approximation so that the graph of the function on the r.h.s. of eq. (4.66) looks qualitatively as the collection of solid curves in fig. 11. For fixed L, the l.h.s. is just proportional to p and hence corresponds to a straight line in fig. 11 passing through the origin. At the points where this line intersects the solid curves, eq. (4.66) is satisfied and a finite volume energy value is found.

If we now vary L, the slope of the straight line in fig. 11 changes and it is then quite obvious how the energy levels evolve as a function of L. Qualitatively, the level pattern obtained in this way is as in fig. 12.

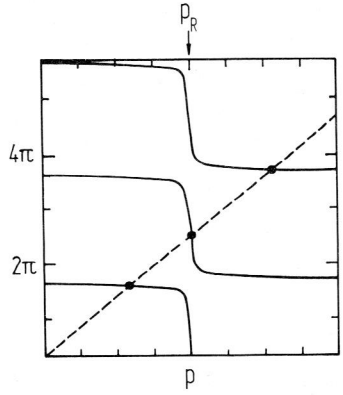


Fig. 11. Graphical solution of eq. (4.66). The full lines represent the r.h.s. of eq. (4.66) as a function of p (in arbitrary units) and the dashed line the l.h.s., for some value of L. The solutions correspond to the intersection points of these curves.

The most characteristic feature of this plot is that at the resonance energy E_R, the individual energy levels develop a plateau whose slope is proportional to Γ/E_R, i.e. for a very narrow resonance, the plateaus are practically horizontal. Thus, for most values of L, one will find a true energy eigenstate in the range $|E - E_R| \leq \Gamma$ and in this sense a narrow resonance behaves like a stable bound state particle. However, as shown by fig. 12, this energy level has a peculiar L-dependence, which does not disappear as the volume is increased, but rather repeats itself in an essentially periodic way. In particular, when L is close to the special values

$$L_n = \frac{2\pi}{p_R}\big(n - \delta_B(p_R)/\pi\big), \qquad n \in \mathrm{Z}, \tag{4.67}$$

one is inbetween the plateaus and there is actually no energy eigenstate which could be unambiguously identified with the resonance there.

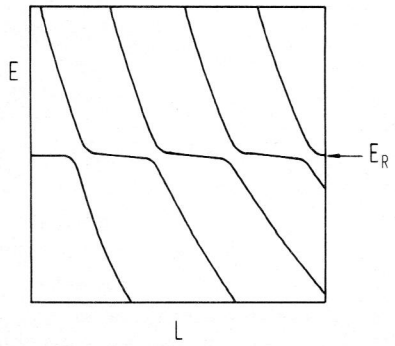


Fig. 12. Qualitative form of the finite volume energy spectrum at energies E near the resonance energy E_R as a function of the volume size L. The different levels correspond to different choices for the quantum number n in eq. (4.66).

I now proceed to discuss resonances in four-dimensional quantum field theories. The theoretical understanding here is still rather incomplete so that I will restrict myself to a few simple remarks. Consider for example a weakly coupled quantum field theory describing two scalar particles σ and π with masses m_σ and m_π such that $2m_\pi < m_\sigma < 3m_\pi$, and suppose the interaction Lagrangian contains a piece which mediates the decay $\sigma \to \pi\pi$ at tree level of perturbation theory. For small coupling then, the σ-particle will be a narrow resonance and a perturbative calculation of the finite volume energy level pattern is feasible. To lowest order of perturbation theory, the eigenstates of the Hamilton operator with zero total momentum and with an energy less than $3m_\pi$ are the vacuum, the one-particle states and the $\pi\pi$-states. Thus, around the resonance mass, the possible finite volume energy values are $E = m_\sigma$ and $E = 2\sqrt{m_\pi^2 + \mathbf{p}^2}$, where $\mathbf{p}$ is quantized in units of $2\pi/L$ as usual.

When we now turn on the interaction, these energy levels are shifted and they then become non-trivial functions of L. For general L, the levels of interest are not degenerate and, repeating the perturbative calculation

presented in section 4.5, one finds that the interaction affects them by an amount proportional to L^{-3}, which is small at large L. An exceptional situation occurs when the volume size L is equal to

$$L_{\mathbf{n}} = \frac{2\pi}{p_R}|\mathbf{n}|, \qquad \mathbf{n} \in \mathrm{Z}^3, \tag{4.68}$$

where $p_R = \frac{1}{2}\sqrt{m_\sigma^2 - 4m_\pi^2}$ denotes the momentum of the pions at the resonance. For these special values of L, the resonance mass happens to be equal to a two-particle energy eigenvalue and to determine the shift in these levels, we must therefore employ degenerate perturbation theory. As a result of all these calculations, one obtains a level pattern which looks very much the same as in fig. 12, with the characteristic plateaus and the "holes" around the singular values (4.68) of L.

From a practical point of view, eq. (4.68) already contains a useful information: it allows one to estimate how wide the resonance plateaus are. For example, in recent numerical simulations of the O(4) non-linear σ-model (see e.g. [87]), a typical value of the σ-mass is 0.5 in lattice units. Since $m_\pi = 0$ in this case, we have $p_R = 0.25$, and eq. (4.68) then yields

$$L/a = 25,\ 36,\ 44,\ 50, \ldots \tag{4.69}$$

for the values of L where strong mixing between the σ-particle state and the two-pion states is expected [4]. Obviously, to avoid large finite size effects on the resonance mass, the lattice sizes L in actual simulations should not come close to any of these singular values. Note that contrary to the one-dimensional case, the spacing between the numbers in the sequence (4.69) shrinks to zero as $L \to \infty$, i.e. the plateaus are less pronounced and the resonance becomes increasingly difficult to observe for large volumes. In any case, the plateaus are reasonably wide in the example considered above and it is likely, therefore, that in most recent calculations of the resonance mass in the O(4) σ-model the "holes" in fig. 12 were avoided.

It should be clear from the discussion so far that the finite volume energy level pattern induced by a resonance deserves further study, both analytical and numerical. In particular, the perturbative arguments need to be worked out in greater detail in a field theoretical setting, and

[4] Since the bound $m_\sigma < 4m_\pi$ is not satisfied in this example, mixing with $2n$-pion states, $n \geq 2$, is possible and this leads to further singular values of L in the range $L/a \geq 43$.

perhaps one is also able to find a way of how to extract the width of a resonance from the energy level shape, assuming this is known from a numerical simulation (cf. [57]).

References

[1] A. Billoire, R. Lacaze, A. Morel, O. Napoly and J. Zinn-Justin, "Field theory on the lattice", Nucl. Phys. B (Proc. Suppl.) 4 (1988).

[2] D. Zwanziger, "Stochastic quantization of gauge fields", in: "Fundamental problems of gauge field theory", Proc. Erice 1985.

[3] J. Zinn-Justin and D. Zwanziger, Nucl. Phys. B295 (1988) 297.

[4] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.

[5] G. 't Hooft, Nucl. Phys. B61 (1973) 455.

[6] T. Balaban and A. Jaffe, "Constructive gauge theory", in: "Fundamental problems of gauge field theory", Proc. Erice 1985.

[7] A. Kupiainen, "Construction of renormalizable and non-renormalizable quantum field theories", Proc. IAMP Conf. Marseille, France, 1986.

[8] M. Lüscher, "Solution of the lattice ϕ^4 theory in 4 dimensions", in: "Non-perturbative quantum field theory", (Cargèse 1987), eds. G. 't Hooft et al., Plenum, to appear.

[9] K.G. Wilson and J. Kogut, Phys. Reports 12 (1974) 75.

[10] K.G. Wilson, "Quantum chromodynamics on a lattice", in: "New developments in quantum field theory and statistical mechanics", Cargèse 1976, Eds. M. Lévy and P. Mitter, Plenum (New York 1977).

[11] M. Lüscher, Commun. Math. Phys. 54 (1977) 238.

[12] M. Creutz, Phys. Rev. D15 (1977) 1128.

[13] K. Osterwalder and E. Seiler, Ann. Phys. (NY) 110 (1978) 440.

[14] E. Seiler, "Gauge theories as a problem of constructive quantum field theory and statistical mechanics", Lecture Notes in Physics, Vol. 159, Springer (Berlin 1982).

[15] K.Fredenhagen, Commun. Math. Phys. 101 (1985) 579.

[16] M.Creutz, Phys. Rev. D35 (1987) 1460.

[17] P. Menotti and A. Pelissetto, Commun. Math. Phys. 113 (1987) 369.

[18] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cim. 1 (1955) 205.

[19] R. Haag, Phys. Rev. 112(1958) 669.

[20] D. Ruelle, Helv. Phys. Acta 35 (1962) 147.

[21] R. Jost, "The general theory of quantized fields", American Mathematical Society, Providence (1965).

[22] J. Glimm and A. Jaffe, "Quantum physics", Springer, New York (1987).

[23] J.C.A. Barata and K. Fredenhagen, "Scattering theory for euclidean lattice fields", in: "Field theory on the lattice", Eds. A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) 4 (1988) 639.

[24] J.C.A. Barata, PhD thesis, in preparation.

[25] L. Fox and I.B. Parker, "Chebyshev polynomials in numerical analysis", Oxford University Press, London (1968).

[26] Y. Hahn and W. Zimmermann, Commun. Math. Phys. 10 (1968) 330.

[27] W. Zimmermann, Commun. Math. Phys. 15 (1969) 208.

[28] J.H. Lowenstein and W. Zimmermann, Nucl. Phys. B86 (1975) 77.

[29] J.H. Lowenstein, Commun. Math. Phys. 47 (1976) 53.

[30] K. Symanzik, "Cutoff dependence in lattice ϕ_4^4 theory", in: Recent developments in gauge theories, eds. G. 't Hooft et al. (Cargèse 1979), Plenum, New York (1980).

[31] K. Symanzik, "Some topics in quantum field theory", in: Mathematical problems in theoretical physics, eds. R. Schrader and D.A. Uhlenbrock, Springer Lecture Notes in Physics, Vol. 153 (1982) 47.

[32] K. Symanzik, Nucl. Phys. B226 (1983) 187 and 205.

[33] M. Lüscher, "Improved lattice gauge theories", in: Critical phenomena, random systems, guage theories, eds. K. Osterwalder and R. Stora, Les Houches session XLIII, Elsevier, Amsterdam (1986).

[34] T. Reisz, Commun. Math. Phys. 116 (1988) 81.

[35] T. Reisz, Commun. Math. Phys. 117 (1988) 79.

[36] T. Reisz, Commun. Math. Phys. 116 (1988) 573 and 117 (1988) 639.

[37] T. Reisz, MPI München preprint, MPI-PAE/PTh 27/88.

[38] M.F.L. Golterman and D. Petcher, Florida preprint, FSU-SCRI-88-19.

[39] G. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42 (1975) 127.

[40] G. Becchi, A. Rouet and R. Stora, Ann. Phys. (NY) 98 (1976) 287.

[41] C. Furrer and A. Lesniewski, Phys. Lett. 148B (1984) 143.

[42] J. Zinn-Justin, Nucl. Phys. B275 [FS17] (1986) 135.

[43] M. Lüscher and P. Weisz, Nucl. Phys. B266 (1986) 309.

[44] M. Lüscher, "On a relation between finite size effects and elastic scattering processes", in: "Progress in gauge field theory", (Cargèse 1983), eds. G. 't Hooft et al., Plenum (New York 1984).

[45] M. Lüscher, Commun. Math. Phys. 104 (1986) 177.

[46] M. Lüscher, Commun. Math. Phys. 105 (1986) 153.

[47] K. Huang and C.N. Yang, Phys. Rev. 105 (1957) 767.

[48] I. Montvay and P. Weisz, Nucl. Phys. B290 [FS20] (1987) 327.

[49] I. Bender, W. Wetzel and B. Berg, Nucl. Phys. B269 (1986) 389.

[50] V. Privman and M.E. Fisher, J.Stat.Phys. 33 (1983) 385.

[51] E. Brézin and J. Zinn-Justin, Nucl. Phys. B257 [FS14] (1985) 867.

[52] G. Münster, to be published.

[53] K. Jansen, J. Jersák, I. Montvay, G. Münster, T. Trappenberg and U. Wolff, Phys. Lett. 213B (1988) 203.

[54] K. Jansen, I. Montvay, G. Münster, T. Trappenberg and U. Wolff, Hamburg preprint, DESY-88-xxx.

[55] H. Leutwyler, Phys. Lett. 189B (1987) 197.

[56] H. Leutwyler and J. Gasser, Nucl. Phys. B307 (1988) 763.

[57] U. Wiese, "Identification of resonance parameters from the finite volume energy spectrum", talk given at the Symposium on Lattice Field Theory, Fermilab (1988), to be published in the proceedings.

[58] G. 't Hooft, Nucl. Phys. B153 (1979) 141.

[59] G. Münster, Nucl. Phys. B180 [FS2] (1981) 23.

[60] M. Lüscher, Nucl. Phys. B180 [FS2] (1981) 317.

[61] J. Cardy, this volume.

[62] M. Lüscher, Nucl. Phys. B219 (1983) 233.

[63] M. Lüscher and G. Münster, Nucl. Phys. B232 (1984) 445.

[64] J. Koller and P. van Baal, Nucl. Phys. B273 (1986) 387.

[65] J. Koller and P. van Baal, Ann. Phys. (NY) 174 (1987) 299.

[66] J. Koller and P. van Baal, Phys. Rev. Lett. 58 (1987) 2511.

[67] J. Koller and P. van Baal, Nucl. Phys. B302 (1988) 1.

[68] J. Koller and P. van Baal, "Analytic results in intermediate volumes for pure SU(2) gauge theory", in: "Field theory on the lattice", Eds. A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) 4 (1988) 47.

[69] J. Koller and P. van Baal, Phys. Rev. Lett. 57 (1986) 2783.

[70] C. Vohwinkel, Phys. Lett. 213B (1988) 54.

[71] P. Weisz and V. Ziemann, Nucl. Phys. B284 (1987) 157.

[72] B.A. Berg, Phys. Lett. 206B (1988) 97.

[73] B.A. Berg, "Glueballs, string tension, tunneling and deconfinement", in: "Field theory on the lattice", Eds. A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) 4 (1988) 6.

[74] B.A. Berg and A. Billoire, Phys. Lett. 166B (1986) 203, (E: 185B (1987) 466).

[75] B.A. Berg, A. Billoire and C. Vohwinkel, Phys. Rev. Lett. 57 (1986) 400.

[76] B. Carpenter, C. Michael and M.J. Teper, Phys. Lett. 198B (1987) 511.

[77] C. Michael and M.J. Teper, Phys. Lett. 199B (1987) 95.

[78] C. Michael, G.A. Tickle and M.J. Teper, Phys. Lett. 207B (1988) 313.

[79] C. Michael, "Glueball masses from large lattices in SU(3) lattice gauge theory" in: "Field theory on the lattice", Eds. A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) 4 (1988) 209.

[80] C. Michael and M.J. Teper, Phys. Lett. 206B (1988) 299.

[81] A.S. Kronfeld, K.J.M. Moriarty and G. Schierholz, preprint, JVNC-87-1.

[82] G. Schierholz, "Glueball masses in SU(3) lattice gauge theory", in: "Field theory on the lattice", Eds. A. Billoire et al., Nucl. Phys. B (Proc. Suppl.) 4 (1988) 11.

[83] APE Collaboration (M. Albanese et al.), Phys. Lett. 192B (1987) 163.

[84] APE Collaboration (M. Albanese et al.), Phys. Lett. 197B (1987) 400.

[85] APE Collaboration (P. Bacilieri et al.), Phys. Lett. 205B (1988) 535.

[86] P. van Baal and A. Kronfeld, "Spectrum of the pure glue theory", talk given at the Symposium on Lattice Field Theory, Fermilab (1988), to be published in the proceedings.

[87] A. Hasenfratz, K. Jansen, J. Jersak, C.B. Lang, T. Neuhaus and H. Yoneyama, "Study of the 4-component Φ^4 model", Jülich preprint, HLRZ-88-02.