# The energy momentum tensor of critical quantum field theories in 1+1 dimensions.

by

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Abstract: We show that the energy momentum tensor of a scaling invariant quantum field theory in two dimensional space time is always a Lie field. Its commutation relations are unique up to two positive real numbers which depend on the dynamics of the model considered.

Recently the Goldstone picture of $P(\phi)_2$ - models has been confirmed [1]. The behaviour of these theories at critical points is however not yet well understood though it is widely believed that some sort of a scaling limit exists. Such a limit is then expected to be a dilation invariant quantum field theory. Thus, this is a field theory satisfying Wightmans axioms [2] and moreover there is a unitary representation $U(\lambda)$, $\lambda > 0$, of the dilatation group such that:

(1) $$U(\lambda)|0\rangle = |0\rangle \quad ; \quad U(\lambda)\,\phi(x)\,U(\lambda)^{-1} = \lambda^d\,\phi(\lambda x)$$

Here, $|0\rangle$ denotes the vacuum state and $\phi$ is any local, covariant field in the theory. $d > 0$ is the dimension of $\phi$.

In this letter we would like to point out that the energy momentum tensor $\theta_{\mu\nu}$ of a dilation invariant theory in 1+1 dimensions is a Lie field [3]. Lie fields occur only rarely in quantum field theory [4] and such a structure is thus very restrictive. For instance, Wightman functions involving $\theta_{\mu\nu}$'s only can be calculated explicitly (see below).

Our assumptions on $\theta_{\mu\nu}$ are the following: it should be a local covariant field with dimension $d = 2$. In addition we suppose that:

(2) $$\theta_{\mu\nu}^{+} = \theta_{\mu\nu} \quad ; \quad \theta_{\mu\nu} = \theta_{\nu\mu} \quad ; \quad \partial^\mu \theta_{\mu\nu} = 0$$

and furthermore

(3) $$\int dx^1 \,[\Theta_{0\mu}(x^0, x^1), \phi(y)] = [P_\mu, \phi(y)] = (-i)\,\partial_\mu \phi(y)$$

for local covariant fields $\phi$.

We now prove that $\Theta_{\mu\nu}$ is traceless: $\Theta^\mu{}_\mu = 0$. To this end consider the Schwinger two point function $S_{\mu\nu\rho\sigma}$ of $\Theta_{\mu\nu}$:

$$S_{\mu\nu\rho\sigma}(x) = \text{analytic continuation of } \langle 0|\,\Theta_{\mu\nu}(x)\,\Theta_{\rho\sigma}(0)\,|0\rangle .$$

This function is real analytic for $x \in \mathbb{R}^2$, $x \neq 0$ and it is covariant under rotations and dilatations:

$$S_{\mu\nu\rho\sigma}(\lambda x) = \lambda^{-4}\, S_{\mu\nu\rho\sigma}(x)$$

Moreover, by (2) and locality we have:

$$S_{\mu\nu\rho\sigma}(x) = S_{\nu\mu\rho\sigma}(x) = S_{\mu\nu\sigma\rho}(x)$$

$$S_{\mu\nu\rho\sigma}(x) = S_{\rho\sigma\mu\nu}(-x) = S_{\rho\sigma\mu\nu}(x)$$

Thus, $S_{\mu\nu\rho\sigma}$ has six independent components and can be written as follows:

(4) $$S_{\mu\nu\rho\sigma} = (x^2)^{-4} \sum_{i=1}^{6} A_i\, T^i_{\mu\nu\rho\sigma} \quad ; \qquad A_i \in \mathbb{C}$$

where

$$T^1_{\mu\nu\rho\sigma} = (x^2)^2 \, g_{\mu\nu} \, g_{\rho\sigma}$$
$$T^2_{\mu\nu\rho\sigma} = (x^2)^2 (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})$$
$$T^3_{\mu\nu\rho\sigma} = x^2 (g_{\mu\nu} x_\rho x_\sigma + g_{\rho\sigma} x_\mu x_\nu)$$
$$T^4_{\mu\nu\rho\sigma} = x_\mu x_\nu x_\rho x_\sigma$$

$$T^5_{\mu\nu\rho\sigma} = x^2 \{ g_{\mu\nu} (x_\rho \epsilon_{\sigma\delta} x^\delta + x_\sigma \epsilon_{\rho\delta} x^\delta) + g_{\rho\sigma} (x_\mu \epsilon_{\nu\delta} x^\delta + x_\nu \epsilon_{\mu\delta} x^\delta) \}$$
$$T^6_{\mu\nu\rho\sigma} = (x_\mu \epsilon_{\nu\delta} x^\delta + x_\nu \epsilon_{\mu\delta} x^\delta) x_\rho x_\sigma + x_\mu x_\nu (x_\rho \epsilon_{\sigma\delta} x^\delta + x_\sigma \epsilon_{\rho\delta} x^\delta)$$

$(\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \; , \quad \epsilon_{01} = +1 \quad ; \quad g_{00} = +g_{11} = +1)$

$T^5$ and $T^6$ are odd under parity and are absent in (4) if parity is conserved.

The continuity equation (2) $\partial^\mu S_{\mu\nu\rho\sigma}(x) = 0$ now fixes the numbers $A_i$ up to two arbitrary constants $A_+ , A_-$ :

$$A_1 = 3A_+ \; , \quad A_2 = -A_+ \; ; \quad A_3 = -4A_+ \; ; \quad A_4 = 8A_+$$
$$A_5 = A_- \; , \quad A_6 = -2A_-$$

Upon inserting these values into eq. (4) we find:

$$S^\mu{}_{\mu\rho\sigma}(x) = S^\mu{}_\mu{}^\rho{}_\rho(x) = 0$$

Hence $\langle 0 | \Theta^\mu{}_\mu(x) \, \Theta^\rho{}_\rho(0) | 0 \rangle$ vanishes which, by the Reeh-Schlieder theorem [2], implies that $\Theta^\mu{}_\mu(x) = 0$.

The fact that $\Theta^\mu{}_\mu(x) = 0$ first of all says that

$\Theta_{\mu\nu}$ has got only two independent components. We may choose them to be

(5) $\Theta_+ \doteq \Theta_{00} + \Theta_{01}$ ; $\Theta_- = \Theta_{00} - \Theta_{01}$

Using once more $\partial^\mu \Theta_{\mu\nu} = 0$ we have

$$\frac{\partial}{\partial x_-} \Theta_+(x_+, x_-) = 0 \quad ; \quad \frac{\partial}{\partial x_+} \Theta_-(x_+, x_-) = 0$$

Here, $x_\pm$ denote the familiar lightcone variables: $x_\pm = x^0 \pm x^1$ *)
Hence $\Theta_+$ ($\Theta_-$) depends on $x_+$ ($x_-$) only.

For clarity we now state the main result of this note in form of a theorem:

<u>Theorem:</u> $\Theta_+(x_+)$ and $\Theta_-(x_-)$ are Lie fields, namely we have:

(6a) $[\Theta_+(x_+), \Theta_+(y_+)] = N_+ \frac{i^3}{12\pi} \delta'''(x_+ - y_+) + 4i\,\delta'(x_+ - y_+)\,\Theta_+(y_+) - 2i\,\delta(x_+ - y_+) \frac{\partial}{\partial y_+} \Theta_+(y_+)$

(6b) $[\Theta_-(x_-), \Theta_-(y_-)] = N_- \frac{i^3}{12\pi} \delta'''(x_- - y_-) + 4i\,\delta'(x_- - y_-)\,\Theta_-(y_-) - 2i\,\delta(x_- - y_-) \frac{\partial}{\partial y_-} \Theta_-(y_-)$

(6c) $[\Theta_+(x_+), \Theta_-(y_-)] = 0$

$N_+$ and $N_-$ are real numbers, $N_\pm \geq 1$. **) In case parity is conserved $N_+ = N_-$.

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*) In Minkowski space we use the metric : $g_{00} = -g_{11} = +1$

**) We have omitted the exceptional case where, say, $N_- = 0$. In this case $\Theta_- = 0$ and the "field theory" is effectively one dimensional.

Before proceeding to the proof of the theorem we would like to make three remarks:

1) The numbers $N_\pm$ in eqs.(6) cannot be removed by rescaling $\Theta_{\mu\nu}$; they are of truly dynamical origin.

2) The two point function of $\Theta_+$ is given by:

$$(7) \quad \langle 0|\Theta_+(x_+)\,\Theta_+(y_+)|0\rangle = \frac{N_+}{4\pi^2}(x_+ - y_+ - i\epsilon)^{-4}$$

Thus, $N_+$ and $N_-$ roughly indicate how large energy momentum fluctuations in the vacuum state are.

3) In case $N_+$ and $N_-$ are natural numbers the algebra (6) can be represented canonically. For the case $N_+ = 1$, $N_- = 0$ respectively $N_+ = 0$, $N_- = 1$ this is done with the help of a hermitian, anticommuting field $\varphi(x)$. Here, $x$ denotes $x_+$ respectively $x_-$. We have:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \frac{1}{2\pi i}(x-y-i\epsilon)^{-1}$$

$$\{\varphi(x), \varphi(y)\} = \delta(x-y)$$

$$\Theta(x) \doteq i :\varphi(x)\frac{\partial}{\partial x}\varphi(x): \doteq i\left\{\varphi(x_1)\frac{\partial}{\partial x_2}\varphi(x_2) - \langle 0|\varphi(x_1)\frac{\partial}{\partial x_2}\varphi(x_2)|0\rangle\right\}_{x_1=x_2=x}$$

and it is straightforward to verify that eqs.(6) hold. Taking tensor products of these two "elementary" representations we obtain a representation of the algebra (6) for all nonnegative integers $N_+$, $N_-$.

This construction suggests that one should call $N_+$ ($N_-$)

the number of (massless) quarks moving from the left to the right (right to left). Quarks and antiquarks are counted seperately. We do not know whether it is possible to have nonintegral quark numbers; we however feel that due to the basically noncompact character of the algebra (6) there is no reason to reject such a possibility.

Proof of the theorem:

1) Eq. (6c) is just a consequence of locality; since for given $x_+$, $y_-$ we can always choose $x, y$ such that $x-y$ is spacelike.

2) To prove (6a) we first simplify our notation by writing $\Theta(x)$ $(x \in \mathbb{R})$ instead of $\Theta_+(x_+)$. From (1) and (3) we have

$$(8) \quad U(\lambda)\,\Theta(x)\,U(\lambda)^{-1} = \lambda^2\,\Theta(\lambda x) \quad ; \quad \int dx\, [\Theta(x), \Theta(y)] = -2i \frac{\partial}{\partial y} \Theta(y)$$

We now investigate the bilocal operator

$$(9) \quad F(z, y) \doteq [\Theta(x), \Theta(y)] \quad ; \quad z = x - y$$

$F(z,y)$ is an operator valued distribution which, by locality, vanishes unless $z = 0$. Let $f(z)$ be a test function which equals 1 in an open neighborhood of zero. Define:

$$(10) \quad O_k(y) \doteq \frac{(-1)^k}{k!} \int dz\, z^k f(z)\, F(z,y) \quad ; \quad k = 0, 1, 2, \ldots$$

These are local operatorvalued distributions. They do not depend on the particular shape of $f(z)$. Therefore

(11) $$U(\lambda)\, O_k(y)\, U(\lambda)^{-1} = \lambda^{3-k}\, O_k(\lambda y)$$

Of course the $O_k(y)$'s are covariant under translations. Hence:

(12) $$\langle 0 |\, O_k^+(x)\, O_k(y) | 0 \rangle = B_k\, (x-y-i\epsilon)^{2k-6} \quad , \quad B_k \in \mathbb{C}$$

This distribution should be positive. As is well known [5], this implies $2k-6 \le 0$. Thus $O_k = 0$ for $k > 3$.

Next we examine the matrixelements of $F(z,y)$. For $|\psi\rangle$, $|\chi\rangle$ in the domain of $\Theta$ the matrixelement $\langle\psi| F(z,y) |\chi\rangle$ is a tempered distribution which is supported on $\{0\} \times \mathbb{R}$. Therefore [5]:

$$\langle\psi| F(z,y) |\chi\rangle = \sum_{\ell=0}^{M} \delta^{(\ell)}(z)\, H_\ell(y) \quad {}^{*)}$$

Of course $H_\ell(y) = \langle\psi| O_\ell(y) |\chi\rangle$ and so $M$ can be set equal to 3. Hence we have the operator equality:

(13) $$[\Theta(x), \Theta(y)] = \sum_{\ell=0}^{3} \delta^{(\ell)}(x-y)\, O_\ell(y)$$

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*) $\delta^{(\ell)}(z)$ denotes the $\ell$'th derivative of $\delta(z)$

We now investigate the consequences of $[\Theta(x),\Theta(y)] = -[\Theta(y),\Theta(x)]$ when inserted into eq. (13):

$$\sum_{\ell=0}^{3} \delta^{(\ell)}(x-y)\, O_\ell(y) = -\sum_{\ell=0}^{3} \delta^{(\ell)}(y-x)\, O_\ell(x)$$

Now $\delta^{(\ell)}(y-x)\, O_\ell(x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \delta^{(k)}(x-y) \frac{\partial^{\ell-k}}{\partial y^{\ell-k}} O_\ell(y)$.

Equating the coefficients of $\delta^{(n)}(x-y)$ we obtain:

(14a) $$O_0(y) = -\sum_{\ell=0}^{3} \frac{\partial^\ell}{\partial y^\ell} O_\ell(y)$$

(14b) $$O_1(y) = \sum_{\ell=1}^{3} \ell \frac{\partial^{\ell-1}}{\partial y^{\ell-1}} O_\ell(y)$$

(14c) $$O_2(y) = -\sum_{\ell=2,3} \frac{\ell(\ell-1)}{2} \frac{\partial^{\ell-2}}{\partial y^{\ell-2}} O_\ell(y)$$

From eq. (12) we observe that $O_3$ is just a constant which can be rewritten as

$$O_3(y) = N \frac{i^3}{12\pi}$$

Now (14c) implies that $O_2(y) = 0$ and (14a) reduces to

$$2O_0(y) = -\frac{\partial}{\partial y} O_1(y)$$

Putting these results together we have:

(15) $$[\Theta(x),\Theta(y)] = N\frac{i^3}{12\pi}\,\delta'''(x-y) + \delta'(x-y)\, O_1(y) - \delta(x-y)\frac{1}{2}\frac{\partial}{\partial y} O_1(y)$$

Remembering eq. (8) shows that we must have

$$\frac{1}{2}\frac{\partial}{\partial y} O_1(y) = 2i \frac{\partial}{\partial y}\Theta(y)$$

which by eqs. (8), (11) and locality of $O_1$ and $\Theta$ integrates uniquely to give $O_1(y) = 4i\,\Theta(y)$. This proves eqs. (6a), (6b).

3) We finally have to prove that $N \geq 1$ (assuming $N \neq 0$).
(*) Notice that the Wightman distributions for $\Theta(x)$ can be calculated using eq. (6). To this end we split $\Theta(x)$ in its positive and negative frequency parts:

$$\tilde{\Theta}(p) \doteq \int dx\, e^{-ipx}\,\Theta(x)$$
$$\Theta^{(-)}(x) \doteq \int_{-\infty}^{0}\frac{dp}{2\pi}\, e^{ipx}\,\tilde{\Theta}(p) \quad ; \quad \Theta^{(+)}(x) = \int_{0}^{\infty}\frac{dp}{2\pi}\, e^{ipx}\,\tilde{\Theta}(p)$$

(16) $$\Theta(x) = \Theta^{(-)}(x) + \Theta^{(+)}(x) \quad ; \quad \Theta^{(-)}(x)\,|0\rangle = 0$$

The commutation relation of $\Theta^{(-)}$ with $\Theta$ is easily calculated:

(17) $$[\Theta^{(-)}(x), \Theta(y)] = \frac{N}{4\pi^2}(x-y-i\epsilon)^{-4} - \frac{2}{\pi}(x-y-i\epsilon)^{-2}\Theta(y) - \frac{1}{\pi}(x-y-i\epsilon)^{-1}\frac{\partial}{\partial y}\Theta(y)$$

The Wightman distributions $W_n(x_1, \dots, x_n) = \langle 0|\,\Theta(x_1)\dots\Theta(x_n)|0\rangle$ can now be computed by inserting expression (16) for $\Theta(x_1)$ and commuting $\Theta^{(-)}(x_1)$ to the right until it reaches the vacuum. This proceedure yields $W_n$ in terms of $W_{n-1}, W_{n-2}$. As is obvious from eq. (17) $W_n$ is a sum of products of integral

(Einschub!)
(*) In order to do this we must first cast the algebra (6) in a more tractable form.

powers of the difference variables $x_i - x_j$. A closer examination shows that $w_n$ is conformally covariant [6]. This is just a reflection of the wellknown fact [7] that dilatation invariance plus the mere existence of an energy momentum tensor already imply that the theory is conformally invariant.

To take advantage of the conformal covariance of $w_n$ we perform a change of variables [8]:

$$x = \mathrm{tg}\,\tau/2 \quad , \quad \tau \in (-\pi, \pi)$$

(18)

$$T(\tau) \doteq (\cos \tau/2)^{-4}\ \theta(x(\tau))$$

Because $x_1 - x_2 - i\epsilon = \dfrac{\sin \frac{1}{2}(\tau_1 - \tau_2 - i\epsilon)}{\cos \tau_1/2 \cdot \cos \tau_2/2}$ $(|\tau_1 - \tau_2| < \pi)$ the Wightman distributions of $T(\tau)$ are sums of products of integral powers of $\sin \frac{1}{2}(\tau_i - \tau_j - i\epsilon)$ $(i<j)$. The factors involving $\cos \tau/2$ cancel by conformal covariance [8]. We thus see that $T(\tau)$ can be extended to a periodic, operator valued distribution on the whole axis $\tau \in (-\infty, \infty)$:

(19) $$T(\tau + 2\pi) = T(\tau)$$

Taking Fourier components

(20) $$X_k \doteq \frac{1}{8} \int_{-\pi}^{\pi} d\tau\, e^{-ik\tau}\, T(\tau) \quad , \quad k \in \mathbb{Z}$$

we arrive at the following algebra:

(21) $[X_\ell, X_k] = \frac{N}{24}\, k(k^2-1)\, \delta_{k+\ell,0} + (k-\ell) X_{k+\ell}$ ; $X_k^+ = X_{-k}$

(22) $X_k |0\rangle = 0$ for $k \leq 1$

Note that the operators $X_k$ can be multiplied at will since they are field operators smeared out with admissable test functions.

The remarkable algebra (21) is well known from dual theories [9] where it is called the Virasoro-algebra (in these theories N is always an integer).

We are now well prepared to prove that $N \geq 1$. This restriction stems from the requirement that the states created from the vacuum by applying a polynomial of $X_k$'s to it should have nonnegative norm. For instance, we have

$$\langle 0| X_k^+ X_k |0\rangle = \langle 0| X_{-k} X_k |0\rangle = \langle 0| [X_{-k}, X_k] |0\rangle = \frac{N}{24}\, k(k^2-1)$$

Thus, N must be positive.

To show that actually the much stronger inequality $N \geq 1$ holds we must calculate the norm of a much more complicated vector, namely:

$$|\psi\rangle = \{8X_9 + 6X_7X_2 + 12X_6X_3 - 8X_2X_5X_2 + 12X_3X_4X_2 - 5X_3X_3X_3\}|0\rangle$$

$|\psi\rangle$ is an eigenstate of $X_0$ and gets annihilated by $X_{-1}$:

$$X_0 |\psi\rangle = 9 \cdot |\psi\rangle \quad , \quad X_{-1} |\psi\rangle = 0$$

(We mention in passing that $|\Psi\rangle$ is a lowest weight vector for the conformal algebra ($= sl(2,\mathbb{R})$) which is spanned by $X_0, X_1, X_{-1}$)
A straightforward but tedious calculation yields the norm of $|\Psi\rangle$:

$$\|\Psi\|^2 = 56\,N(N-1)(5N+44)$$

Hence, $\|\Psi\|^2 < 0$ for $0 < N < 1$ and so $N \geq 1$ or $N = 0$. ∎

We would like to make two concluding remarks:

1) We do not have a proof that the representation of the Virasoro algebra (21), which is specified by (22), is really a representation in a positive metric Hilbert space also in the case, when N is non integral (and, of course, $N \geq 1$).

2) In case the underlying field theory contains other fields than just $\theta_{\mu\nu}$ there will be representations of (21) that are inequivalent to the "vacuum representation" discussed so far. Because $X_0 \geq 0$ [8], we expect them to be representations with a lowest weight vector $|\Omega\rangle$:

$$X_k|\Omega\rangle = 0 \quad \text{for} \quad k \leq -1 \quad , \quad X_0|\Omega\rangle = \lambda|\Omega\rangle \; ; \; \lambda > 0 .$$

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