Topological fields in Yang-Mills gauge theories

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1. Introduction

In this note it is shown that the Chern forms are the only local topological fields that can be constructed from a gauge potential $A_\mu^a(x)$ in n dimensions. It is possible to prove this using the BRS symmetry and the so-called descent equations, which have been introduced to determine the general structure of the non-abelian gauge anomaly (see refs. [1–3] for a review and a list of references). The argument presented here does not make use of this powerful machinery and seems to be somewhat simpler although the basic ideas are rather similar.

By definition a topological field is a gauge-invariant polynomial $q(x)$ in the gauge potential $A_\mu^a(x)$ and its derivatives such that

$$\int \mathrm{d}^n x \, \delta q(x) = 0 \tag{1.1}$$

for all variations $\delta A_\mu^a(x)$ of the gauge field with compact support. A trivial case are the fields of the form

$$q(x) = \partial_\mu k_\mu(x), \tag{1.2}$$

where $k_\mu(x)$ is a gauge-invariant local current. One only requires the classification of all topological fields modulo such terms and thus has a special case of a cohomology problem in which the gauge symmetry plays an important rôle.

2. Chern forms

Examples of topological fields in any dimension are the Chern forms

$$c_{\mu_1\ldots\mu_{2r}} t^{a_1\ldots a_r} F^{a_1}_{\mu_1\mu_2}(x)\ldots F^{a_r}_{\mu_{2r-1}\mu_{2r}}(x), \tag{2.1}$$

where $c_{\mu_1\ldots\mu_{2r}}$ is a totally anti-symmetric constant tensor and $t^{a_1\cdots a_r}$ a tensor which is invariant under the adjoint action of the gauge group G (see appendix A for a summary of notations). Using the Bianchi identity and the G-invariance of the expression it is in fact easy to show that these fields are topological in the sense explained above.

For any fixed dimension and gauge group G there are at most a finite number of linearly independent Chern forms. To see this first note that there are no totally anti-symmetric tensors $c_{\mu_1\ldots\mu_{2r}}$ with more than n indices. As far as the other tensor $t^{a_1\cdots a_r}$ is concerned, we may assume that it is totally symmetric since it is contracted with a symmetric expression. The classification of these tensors rests on the observation that

$$P(X) = t^{a_1\ldots a_r} X^{a_1}\ldots X^{a_r}, \qquad X = X^a T^a, \tag{2.2}$$

defines a homogeneous G-invariant polynomial in the components X^a of the Lie algebra element X. Conversely any such polynomial corresponds to a unique tensor $t^{a_1\cdots a_r}$ with the required properties. Note that the linear space of all invariant polynomials is closed under multiplication and thus forms an algebra.

The Lie algebra of $\mathrm{SU}(N)$ can be taken to be the space of complex anti-hermitean $N \times N$ matrices with vanishing trace. In this case the algebra of invariant polynomials is generated by the polynomials

$$\mathrm{Tr}\{X^2\},\ \mathrm{Tr}\{X^3\},\ldots,\ \mathrm{Tr}\{X^N\} \tag{2.3}$$

(see ref. [4], §2.1, for example). Similar results can be established for the other classical groups and it is then straightforward to produce a complete list of linearly independent Chern forms.

For illustration let us consider an $\mathrm{SU}(N)$ theory in $n = 6$ dimensions. If we choose the group generators to be orthonormal,

$$\mathrm{Tr}\{T^a T^b\} = -\tfrac{1}{2}\delta^{ab}, \tag{2.4}$$

the only symmetric invariant tensor apart from 1 and δ^{ab} that can contribute is

$$d^{abc} = 2i \,\mathrm{Tr}\{T^a(T^bT^c + T^cT^b)\} \tag{2.5}$$

and the general linear combination of Chern forms is thus given by

$$\alpha + \beta_{\mu_1\mu_2}\epsilon_{\mu_1\ldots\mu_6}F^a_{\mu_3\mu_4}F^a_{\mu_5\mu_6} + \gamma\epsilon_{\mu_1\ldots\mu_6}d^{abc}F^a_{\mu_1\mu_2}F^b_{\mu_3\mu_4}F^c_{\mu_5\mu_6}, \tag{2.6}$$

where α, $\beta_{\mu\nu}$ and γ are arbitrary constant coefficients.

3. Statement of result

As already mentioned we are considering gauge-invariant local fields $q(x)$ that are polynomials in the gauge potential $A^a_\mu(x)$ and its derivatives. Such a field is referred to as topological if eq. (1.1) holds for any local deformation $\delta A^a_\mu(x)$ of the gauge potential. This is equivalent to the requirement that

$$\sum_{|\alpha|\geq 0}(-1)^{|\alpha|}\partial^\alpha \frac{\partial q(x)}{\partial[\partial^\alpha A^a_\mu(x)]} = 0, \tag{3.1}$$

where a multi-index notation has been used (as explained in appendix A) and the derivatives with respect to the gauge field and its derivatives are to be taken in the obvious way after expanding $q(x)$ in a sum of products of these fields.

The main result of this note is now summarized by

Theorem 3.1. *Any topological field $q(x)$ is of the form*

$$q(x) = c(x) + \partial_\mu k_\mu(x), \tag{3.2}$$

where $c(x)$ is a sum of Chern forms and $k_\mu(x)$ a gauge-invariant polynomial in the gauge potential and its derivatives.

The proof of this theorem is somewhat involved and will be broken up into several steps. We first reduce the problem to the abelian case and then proceed essentially along the lines of ref. [5].

4. Reduction to the abelian case

Any polynomial in the gauge potential and its derivatives can be written in a unique way as a sum of homogeneous polynomials of increasing degree. If the polynomial is gauge-invariant, it is then immediately clear that the homogeneous part of lowest degree is invariant under global gauge transformations,

$$A^a_\mu(x) \to (\mathrm{Ad}\, g)^{ab} A^b_\mu(x), \qquad g \in G, \tag{4.1}$$

and the "abelian" transformations

$$A^a_\mu(x) \to A^a_\mu(x) + \partial_\mu \omega^a(x). \tag{4.2}$$

Local fields that are invariant under these transformations (collectively referred to as *linearized gauge transformations*) are easily constructed by forming G-invariant products of the linearized field tensor

$$\check{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \tag{4.3}$$

and its derivatives. In fact, as asserted by the following lemma, all invariant fields are of this type.

Lemma 4.1. *Any local field $\check{p}(x)$ which is a polynomial in the gauge potential $A^a_\mu(x)$ and its derivatives and which is invariant under the group of linearized gauge transformations is equal to a G-invariant polynomial in $\check{F}^a_{\mu\nu}(x)$ and its derivatives.*

Proof: Let y be any reference point and define the gauge transformation function

$$\omega^a(x) = -\int_0^1 \mathrm{d}t\,(x-y)_\lambda A^a_\lambda(y + t(x-y)). \tag{4.4}$$

It is then straightforward to establish the identity

$$\widetilde{A}^a_\mu(x) \equiv A^a_\mu(x) + \partial_\mu \omega^a(x) = \int_0^1 \mathrm{d}t\, t(x-y)_\lambda \check{F}^a_{\lambda\mu}(y + t(x-y)) \tag{4.5}$$

and after expanding in a Taylor series around $x = y$ one obtains

$$\widetilde{A}^a_\mu(y) = 0, \qquad \partial_{\nu_1} \ldots \partial_{\nu_r} \widetilde{A}^a_\mu(y) = \frac{1}{r+1} \sum_{k=1}^{r} \partial_{\nu_1} \ldots \hat{\partial}_{\nu_k} \ldots \partial_{\nu_r} \check{F}^a_{\nu_k \mu}(y), \tag{4.6}$$

where the notation $\hat{\partial}_{\nu_k}$ implies that the derivative ∂_{ν_k} should be omitted.

The field $\check{p}(x)$ is a polynomial in the gauge potential and its derivatives which is invariant under linearized gauge transformations. We may, therefore, replace $A^a_\mu(x)$ by the transformed field (4.5) in this expression and in view of eq. (4.6) it is then immediately clear that $\check{p}(y)$ is equal to a G-invariant polynomial in the linearized field tensor $\check{F}^a_{\mu\nu}(y)$ and its derivatives. □

Some further clarification of the relation between fields that are invariant under linearized gauge transformations and the gauge-invariant fields is provided by

Lemma 4.2. *Any homogeneous polynomial $\check{p}(x)$ in the gauge potential $A^a_\mu(x)$ and its derivatives, which is invariant under the group of linearized gauge transformations, coincides with the lowest-degree homogeneous part of a gauge-invariant polynomial $p(x)$.*

Proof: Lemma 4.1 implies that $\check{p}(x)$ is equal to a sum of monomials of the type

$$t^{a_1\ldots a_r}\partial^{\alpha_1}\check{F}^{a_1}_{\mu_1\nu_1}(x)\ldots\partial^{\alpha_r}\check{F}^{a_r}_{\mu_r\nu_r}(x), \tag{4.7}$$

where we have again made use of the multi-index notation for partial derivatives (cf. appendix A). The sum of all these terms is G-invariant, i.e. if we substitute

$$\check{F}^a_{\mu\nu} \to (\mathrm{Ad}\, g)^{ab}\check{F}^b_{\mu\nu}, \qquad g \in G, \tag{4.8}$$

the expression will not change. One cannot conclude from this that the tensors $t^{a_1\ldots a_r}$ are G-invariant since there are algebraic dependencies among the derivatives of the linearized field tensor. From the above one however infers that the tensors may be replaced by

$$\bar{t}^{a_1\ldots a_r} = \int_G \mathrm{d}g\, (\mathrm{Ad}\, g)^{a_1b_1}\ldots(\mathrm{Ad}\, g)^{a_rb_r}t^{b_1\ldots b_r}, \tag{4.9}$$

without changing the sum of all terms, $\mathrm{d}g$ being the normalized invariant measure on G (the integral is well-defined since G is compact).

Evidently the new tensors $\bar{t}^{a_1\ldots a_r}$ are G-invariant and a gauge-invariant polynomial $p(x)$ may thus be defined by substituting

$$\bar{t}^{a_1\ldots a_r}D^{\alpha_1}F^{a_1}_{\mu_1\nu_1}(x)\ldots D^{\alpha_r}F^{a_r}_{\mu_r\nu_r}(x), \tag{4.10}$$

for the terms (4.7), where D^{α_k} is the totally symmetrized product of covariant derivatives corresponding to the multi-index α_k. Since the degree r is the same for all these terms, it is then obvious that the lowest-degree homogeneous part of $p(x)$ is equal to $\check{p}(x)$. $\square$

The following theorem may be regarded as the abelian version of theorem 3.1. We shall prove it in sect. 6 and shall then be able to establish theorem 3.1 with relatively little effort by setting up a recursion over the degree of the homogeneous parts of the topological field (sect. 7).

Theorem 4.3. *The lowest-order homogeneous part of any topological field $q(x)$ is of the form*

$$\check{q}(x) = \check{c}(x) + \partial_\mu \check{k}_\mu(x), \tag{4.11}$$

where $\check{c}(x)$ and $\check{k}_\mu(x)$ are the lowest-order homogeneous parts of a sum $c(x)$ of Chern forms and of a current $k_\mu(x)$, which is a gauge-invariant polynomial in the gauge potential $A^a_\mu(x)$ and its derivatives.

5. Poincaré lemma

In its widely known form the Poincaré lemma states that a closed k-form on $\mathbb{R}^n$ is exact if $k < n$. The aim here is to establish another version of the lemma which applies to partial differential operators acting on differential forms. For the proof of theorem 4.3 this will turn out to be a most effective tool.

5.1 Differential forms

Following standard notations the general k-form on $\mathbb{R}^n$ is given by

$$f(x) = \frac{1}{k!} f_{\mu_1 \ldots \mu_k}(x) \mathrm{d}x_{\mu_1} \ldots \mathrm{d}x_{\mu_k}, \tag{5.1}$$

where $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate a Grassmann algebra. We shall always take it for granted that the coefficient functions $f_{\mu_1 \ldots \mu_k}(x)$ are smooth and denote the space of all these forms by Ω_k. A special case is Ω_0 which coincides with the space of smooth functions $f(x)$ on $\mathbb{R}^n$. For notational convenience it is useful to set $\Omega_k = \{0\}$ if $k < 0$ or $k > n$.

The exterior differential operator d acts on k-forms according to

$$\mathrm{d}f(x) = \frac{1}{k!} \partial_\mu f_{\mu_1 \ldots \mu_k}(x) \, \mathrm{d}x_\mu \mathrm{d}x_{\mu_1} \ldots \mathrm{d}x_{\mu_k} \tag{5.2}$$

and thus maps Ω_k to Ω_{k+1}. In particular, $\mathrm{d}f(x) = 0$ if $f(x)$ is an n-form and the same is true for any $k > n$.

The associated divergence operator $\mathrm{d}^* : \Omega_k \to \Omega_{k-1}$ is defined similarly through

$$\mathrm{d}^* f(x) = \frac{1}{(k-1)!}\, \partial_\mu f_{\mu\mu_2\ldots\mu_k}(x)\mathrm{d}x_{\mu_2}\ldots\mathrm{d}x_{\mu_k} \tag{5.3}$$

if $0 < k \leq n$ and $\mathrm{d}^* f(x) = 0$ in all other cases. With respect to the natural scalar product for tensor fields, d^* is equal to minus the adjoint of d.

The general form of a differential operator $L : \Omega_l \to \Omega_k$ is

$$Lf(x) = \frac{1}{k!\,l!} \sum_{0\leq|\alpha|\leq r} \mathrm{d}x_{\mu_1}\ldots\mathrm{d}x_{\mu_k} L^{\alpha}_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}(x)\partial^\alpha f_{\nu_1\ldots\nu_l}(x), \tag{5.4}$$

where α is a multi-index and r the degree of L. The coefficients $L^{\alpha}_{\mu_1\ldots\mu_k,\nu_1\ldots\nu_l}$ may be assumed to be totally anti-symmetric in the first group $\mu_1,\ldots,\mu_k$ and the second group $\nu_1,\ldots,\nu_l$ of indices separately. d and d^* are simple examples of such operators and it is evident that operator products like $\mathrm{d}L$ and $L\mathrm{d}$ are also of this type. More generally any product of L with partial derivatives ∂_σ and the differentials $\mathrm{d}x_\sigma$ results in an operator of the form (5.4) after reordering of the factors.

5.2 Poincaré lemma for differential operators

Since $\mathrm{d}^2 = 0$ the equation $\mathrm{d}L = 0$ is satisfied by all operators of the form $L = \mathrm{d}K$. We would now like to show that these are all solutions if $k < n$ and that K can be constructed locally from the coefficients of L.

To this end some preparation is still needed. We first establish the following basic lemma, which allows one to "pull out" a partial derivative from an arbitrary differential operator.

Lemma 5.1. *Let $L : \Omega_l \mapsto \Omega_k$ be a differential operator as described above and choose a fixed direction μ. Then there exist two other operators S and R of the same type, with S having zero degree in ∂_μ, such that*

$$L = S + \partial_\mu R. \tag{5.5}$$

This decomposition is unique and the coefficients of S and R are integer linear combinations of the coefficients of L and their derivatives.

Proof: Starting from the general expression for L, eq. (5.4), we have

$$L = \sum_{q=0}^{r} T^q \partial_\mu^q, \tag{5.6}$$

where $T^q : \Omega_l \mapsto \Omega_k$ are differential operators of zero degree in the derivative ∂_μ. Now if we define S and R through

$$S = \sum_{q=0}^{r} (-1)^q [\partial_\mu^q T^q], \tag{5.7}$$

$$R = \sum_{q=1}^{r} \sum_{p=0}^{q-1} (-1)^p [\partial_\mu^p T^q] \partial_\mu^{q-p-1}, \tag{5.8}$$

it is straightforward to verify that $L = S + \partial_\mu R$. In these equations it is understood that the derivatives ∂_μ^s inside the square brackets apply to the coefficients of T^q only and not also to the form $f(x)$ on which these operators act.

The uniqueness of the decomposition can be established by writing R as a polynomial in ∂_μ in the same way as L. After substituting

$$\partial_\mu R = [\partial_\mu, R] + R\partial_\mu, \tag{5.9}$$

the terms in eq. (5.5) with the same power of ∂_μ have to match and this leads to a simple ladder of equations which determines the coefficients of R. $\square$

For any given operator L we can now generate a descending sequence of operators L_m, R_m and Z_m such that

$$L_n = L, \tag{5.10}$$

$$L_m = (L_{m-1} + \partial_m R_{m-1})\,\mathrm{d}x_m + Z_m, \quad m = n, n-1, \ldots, 1. \tag{5.11}$$

In the recursion step, eq. (5.11), one decomposes L_m in the part proportional to $\mathrm{d}x_m$ and the remainder Z_m and applies lemma 5.1 to the first term. The sequence is thus well-defined, with the coefficients of all operators being integer linear combinations of the coefficients of L and their derivatives. By construction L_m, R_m and Z_m are differential operators mapping l-forms to $(k-n+m)$-forms which are independent of $\mathrm{d}x_{m+1}, \mathrm{d}x_{m+2}, \ldots, \mathrm{d}x_n$. Moreover L_m has zero degree in $\partial_{m+1}, \partial_{m+2}, \ldots, \partial_n$.

We are now in a position to state the Poincaré lemma for differential operators. Compared to the classical lemma for differential forms, an important difference is that the "potential" K is obtained locally from L. In particular, if the coefficients of L are gauge-invariant polynomials in the gauge field and its derivatives the same will be true for the coefficients of K.

Lemma 5.2. *If* $L : \Omega_l \mapsto \Omega_k$ *is any differential operator satisfying* $\mathrm{d}L = 0$ *we have*

$$L = \delta_{kn} L_0 \mathrm{d}x_1 \ldots \mathrm{d}x_n + \mathrm{d}K, \tag{5.12}$$

$$K = \sum_{m=n-k}^{n-1} (-1)^{k-n+m} R_m \mathrm{d}x_{m+2} \ldots \mathrm{d}x_n, \tag{5.13}$$

where L_0 *and* R_m *are obtained by solving the recursion (5.10),(5.11).*

Proof: We first show recursively that

$$\mathrm{d}L_m \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n = 0. \tag{5.14}$$

For $m = n$ this is trivially the case. Now if we assume that the equation holds for some fixed $m \leq n$, the recursion (5.11) implies

$$\bar{\mathrm{d}}(L_{m-1} + \partial_m R_{m-1})\mathrm{d}x_m + (\bar{\mathrm{d}} + \mathrm{d}x_m \partial_m) Z_m = 0, \qquad \bar{\mathrm{d}} \equiv \sum_{\mu=1}^{m-1} \mathrm{d}x_\mu \partial_\mu. \tag{5.15}$$

In particular, the term proportional to $\mathrm{d}x_m$ has to vanish,

$$\bar{\mathrm{d}} L_{m-1} + \partial_m \left\{ \bar{\mathrm{d}} R_{m-1} + (-1)^{k-n+m} Z_m \right\} = 0, \tag{5.16}$$

and since L_{m-1} has zero degree in ∂_m it follows from this and lemma 5.1 that $\bar{\mathrm{d}} L_{m-1} = 0$ and thus $\mathrm{d}L_{m-1}\mathrm{d}x_m \ldots \mathrm{d}x_n = 0$.

By induction this proves eq. (5.14). Moreover from eq. (5.16) one infers that

$$\bar{\mathrm{d}} R_{m-1} + (-1)^{k-n+m} Z_m = 0 \tag{5.17}$$

for all $m = n, n-1, \ldots, 1$. Together with eq. (5.11) this leads to the identity

$$L_m \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n = \left\{ L_{m-1} \mathrm{d}x_m - (-1)^{k-n+m} \mathrm{d}R_{m-1} \right\} \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \tag{5.18}$$

and eqs. (5.12) and (5.13) are now obtained straightforwardly. □

Equivalent forms of the Poincaré lemma can be derived for operators L satisfying $\mathrm{d}^* L = 0$ or one of the equations $L\mathrm{d} = 0$ or $L\mathrm{d}^* = 0$. In each case a zero degree term L_0 may appear if the form degrees k or l are such that the equation is satisfied for any L.

6. Proof of theorem 4.3

The strategy of the proof is to extract powers of the linearized field tensor $\check{F}^a_{\mu\nu}(x)$ from $\check{q}(x)$ in a recursive manner. One is generating tensor fields of increasing rank along the way and it is crucial to employ a compact notation to keep the argumentation transparent.

6.1 Notational conventions

In this section the term "local field" stands for a polynomial in the gauge potential $A^a_\mu(x)$ and its derivatives which is invariant under abelian gauge transformations [eq. (4.2)] and which transforms as a tensor under the action (4.1) of G. We also admit local fields that are p-forms and the linear space of all these fields is denoted by Λ_p. The linearized field tensor

$$\check{F}^a(x) = \tfrac{1}{2}\check{F}^a_{\mu\nu}(x)\mathrm{d}x_\mu \mathrm{d}x_\nu \tag{6.1}$$

is an element of Λ_2, for example, but the gauge field 1-form

$$A^a(x) = A^a_\mu(x)\mathrm{d}x_\mu \tag{6.2}$$

is not contained in Λ_1 since it is not invariant under abelian gauge transformations.

For any field $\theta \in \Lambda_p$ we define an operator $\hat{\theta} : \Omega_k \to \Omega_{p-k}$ through

$$\hat{\theta}f(x) = \frac{1}{(p-k)!\,k!}\mathrm{d}x_{\mu_1}\ldots\mathrm{d}x_{\mu_{p-k}}\theta_{\mu_1\ldots\mu_{p-k}\nu_1\ldots\nu_k}(x)f_{\nu_1\ldots\nu_k}(x) \tag{6.3}$$

for all k in the range $0 \leq k \leq p$ and $\hat{\theta}f(x) = 0$ in all other cases. Evidently $\hat{\theta}$ may be regarded as a differential operator of zero degree and it thus fits into the general framework of sect. 5. If $\theta(x)$ has vanishing divergence, $\mathrm{d}^*\theta(x) = 0$, it is straightforward to show that

$$\mathrm{d}^*\hat{\theta}f(x) = (-1)^{p-k-1}\hat{\theta}\mathrm{d}f(x) \tag{6.4}$$

for all k-forms $f(x)$.

6.2 Recursion

The recursion which will be set up is based on two lemmas which are stated and proved below. The first lemma provides the starting point of the recursion and the second allows one to extract the next power of the linearized field tensor from the current expression. For clarity the lemmas are formulated in greater generality than would be necessary for the proof of theorem 4.3. Their application will be discussed in the next subsection.

Lemma 6.1. *Let $\phi \in \Lambda_0$ be a given field satisfying $\int \mathrm{d}^n x\, \delta\phi(x) = 0$ for any variation $\delta A^a_\mu(x)$ of the gauge potential with compact support. Then there exist local fields $\theta^a \in \Lambda_2$ and $\omega \in \Lambda_1$ such that*

$$\mathrm{d}^*\theta^a(x) = 0, \tag{6.5}$$

$$\phi(x) = c + \hat{\theta}^a \check{F}^a(x) + \mathrm{d}^*\omega(x), \tag{6.6}$$

where c is a constant.

Proof: Without loss of generality we may assume that ϕ is a homogeneous polynomial in the gauge potential and its derivatives, viz.

$$\sum_{|\alpha|\geq 0} \partial^\alpha A^a_\mu(x) \frac{\partial\phi(x)}{\partial[\partial^\alpha A^a_\mu(x)]} = h\phi(x) \tag{6.7}$$

for some integer $h \geq 0$. If $h = 0$ the statement made in the lemma is trivial and it thus remains to consider the case where h is positive.

If we introduce a differential operator $L^a : \Omega_1 \to \Omega_0$ through

$$L^a f(x) = \sum_{|\alpha|\geq 0} \frac{\partial\phi(x)}{\partial[\partial^\alpha A^a_\mu(x)]} \partial^\alpha f_\mu(x), \tag{6.8}$$

it is evident from the above that

$$h\phi(x) = L^a A^a(x), \tag{6.9}$$

$$\delta\phi(x) = L^a \delta A^a(x). \tag{6.10}$$

The second equation and the fact that ϕ is invariant under abelian gauge transformations imply $L^a \mathrm{d} = 0$. Recalling the Poincaré lemma we may conclude from this that there exists another operator $K^a : \Omega_2 \to \Omega_0$ such that $L^a = K^a \mathrm{d}$. Moreover since $\mathrm{d}^* K^a = 0$ we have $K^a = K^a_0 + \mathrm{d}^* H^a$ where the first term has zero degree. In other words, there exists a field $\theta^a \in \Lambda_2$ such that $K^a_0 = h\hat{\theta}^a$.

Putting everything together, eqs. (6.9) and (6.10) become

$$h\phi(x) = h\hat{\theta}^a \check{F}^a(x) + \mathrm{d}^* H^a \check{F}^a(x), \tag{6.11}$$

$$\delta\phi(x) = h\hat{\theta}^a \mathrm{d}\delta A^a(x) + \mathrm{d}^* H^a \mathrm{d}\delta A^a(x). \tag{6.12}$$

The first of these equations assumes the form (6.6) (with $c = 0$) if we set

$$H^a \check{F}^a(x) = h\omega(x). \tag{6.13}$$

An important point to note here is that the coefficients of all the operators and fields that we have introduced above are local fields in the sense defined in subsection 6.1. In particular, $\omega(x)$ is invariant under linearized gauge transformations.

We finally need to show that $\mathrm{d}^*\theta^a(x) = 0$. To this end we integrate eq. (6.12) over $\mathbb{R}^n$ and note that the left-hand side and the second term on the other side do not contribute to the integral. This leads to the identity

$$\int \mathrm{d}^n x\, \hat{\theta}^a \mathrm{d}\delta A^a(x) = 0 \tag{6.14}$$

and after performing a partial integration one concludes from this that $\theta^a(x)$ has to have vanishing divergence. $\square$

Lemma 6.2. *Let $\phi^{a_1\cdots a_r} \in \Lambda_p$, $p \geq 1$, be a given field satisfying $\mathrm{d}^*\phi^{a_1\cdots a_r}(x) = 0$. Then there exist local fields $\theta^{a_1\cdots a_{r+1}} \in \Lambda_{p+2}$ and $\omega^{a_1\cdots a_r} \in \Lambda_{p+1}$ such that*

$$\mathrm{d}^*\theta^{a_1\cdots a_{r+1}}(x) = 0, \tag{6.15}$$

$$\phi^{a_1\cdots a_r}(x) = c^{a_1\cdots a_r} + \hat{\theta}^{a_1\cdots a_{r+1}} \check{F}^{a_{r+1}}(x) + \mathrm{d}^*\omega^{a_1\cdots a_r}(x), \tag{6.16}$$

where $c^{a_1\cdots a_r}$ is a constant G-invariant p-form.

Proof: For simplicity we omit the indices $a_1 \ldots a_r$ in the following since their presence is irrelevant for our argumentation and would not interfere in any way. As in the proof of lemma 6.1 it suffices to consider the case where $\phi(x)$ is a homogeneous polynomial in the gauge potential and its derivatives of degree $h > 0$. If we define the operator $L^a : \Omega_1 \to \Omega_p$ as before [eq. (6.8)], it is then immediately clear that eqs. (6.9) and (6.10) remain valid in the present context.

From the second equation and the properties of ϕ it now follows that

$$\mathrm{d}^* L^a = L^a \mathrm{d} = 0. \tag{6.17}$$

Applying the Poincaré lemma two times, L^a may thus be decomposed according to

$$L^a = \mathrm{d}^* K^a, \qquad K^a : \Omega_1 \to \Omega_{p+1}, \tag{6.18}$$

$$K^a \mathrm{d} = \mathrm{d}^* H^a, \qquad H^a : \Omega_0 \to \Omega_{p+2}. \tag{6.19}$$

Since $H^a \mathrm{d} = 0$ another application of the lemma yields

$$H^a = H_0^a + R^a \mathrm{d}, \qquad R^a : \Omega_1 \to \Omega_{p+2}, \tag{6.20}$$

where H_0^a has zero degree, i.e. there exists a field $\theta^a \in \Lambda_{p+2}$ such that $H_0^a = -h\hat{\theta}^a$. Moreover we may assume that $R^a = 0$ since this term can be removed by replacing K^a through $K^a + \mathrm{d}^* R^a$.

If we apply eq. (6.19) to a constant 0-form, the left-hand side vanishes and the other side is proportional to $\mathrm{d}^*\theta^a(x)$. This shows that $\mathrm{d}^*\theta^a(x) = 0$. In particular, eq. (6.4) applies and eq. (6.19) assumes the form

$$K^a \mathrm{d} = (-1)^p h \hat{\theta}^a \mathrm{d}. \tag{6.21}$$

Invoking the Poincaré lemma once more, this implies

$$K^a = (-1)^p h \hat{\theta}^a + Q^a \mathrm{d}, \qquad Q^a : \Omega_2 \to \Omega_{p+1}. \tag{6.22}$$

The operator L^a is thus given by

$$L^a = h \hat{\theta}^a \mathrm{d} + \mathrm{d}^* Q^a \mathrm{d}, \tag{6.23}$$

where we have again made use of eq. (6.4). Together with eq. (6.9) this proves eq. (6.16) if we set $Q^a \check{F}^a(x) = h\omega(x)$ and $c = 0$. □

6.3 Final steps

We now complete the proof of theorem 4.3 by combining the lemmas established above. First note that

$$\check{q}(x) = c + \hat{\theta}^a \check{F}^a(x) + \mathrm{d}^*\omega(x), \tag{6.24}$$

since $q(x)$ is a topological field and $\check{q}(x)$ thus fulfills the premises of lemma 6.1. The fields $\theta^a(x)$ and $\omega(x)$ are the first elements of a sequence of fields

$$\theta^{a_1 \ldots a_r} \in \Lambda_{2r}, \qquad \omega^{a_1 \ldots a_{r-1}} \in \Lambda_{2r-1}, \tag{6.25}$$

$$\mathrm{d}^*\theta^{a_1 \ldots a_r}(x) = 0, \qquad r \geq 1, \tag{6.26}$$

which may be generated by repeated application of lemma 6.2. By construction we have

$$\theta^{a_1\cdots a_r}(x) = c^{a_1\cdots a_r} + \hat{\theta}^{a_1\cdots a_{r+1}}\check{F}^{a_{r+1}}(x) + \mathrm{d}^*\omega^{a_1\cdots a_r}(x), \tag{6.27}$$

where $c^{a_1\cdots a_r}$ are constant G-invariant p-forms. Moreover since $\check{q}(x)$ is a homogenous polynomials of degree $h \geq 0$ in the gauge potential and its derivatives, we may assume that $\theta^{a_1\cdots a_r}(x)$ and $\omega^{a_1\cdots a_r}(x)$ are also homogenous with degree $h - r$. In particular, the constants $c^{a_1\cdots a_r}$ have to be equal to zero except possibly for $r = h$.

Starting from eq. (6.24) we can now eliminate the fields $\theta^{a_1\cdots a_r}(x)$ recursively using eq. (6.27). The recursion terminates if $r > h$ or $2r > n$, because $\theta^{a_1\cdots a_r}(x) = 0$ for such values of r. In each step one generates a term of the form

$$\partial_\mu \omega^{a_1\ldots a_r}_{\mu\mu_1\ldots\mu_{2r}}(x)\check{F}^{a_1}_{\mu_1\mu_2}(x)\ldots\check{F}^{a_1}_{\mu_{2r-1}\mu_{2r}}(x) \tag{6.28}$$

and if $r = h$ also a second term proportional to

$$c^{a_1\ldots a_r}_{\mu_1\ldots\mu_{2r}}\check{F}^{a_1}_{\mu_1\mu_2}(x)\ldots\check{F}^{a_1}_{\mu_{2r-1}\mu_{2r}}(x). \tag{6.29}$$

Evidently the latter is a equal to the lowest-order homogeneous part of a sum of Chern forms. The other term may be rewritten in the form

$$\partial_\mu \left\{\omega^{a_1\ldots a_r}_{\mu\mu_1\ldots\mu_{2r}}(x)\check{F}^{a_1}_{\mu_1\mu_2}(x)\ldots\check{F}^{a_1}_{\mu_{2r-1}\mu_{2r}}(x)\right\} \tag{6.30}$$

as a result of the Bianchi identity and the fact that the components of $\omega^{a_1\cdots a_r}(x)$ are totally anti-symmetric in the Lorentz indices. The sum of all these terms is thus equal to the divergence of a current $\check{k}_\mu(x)$ which is homogenous of degree h in the gauge potential and its derivatives. Moreover, since it $\check{k}_\mu(x)$ is invariant under linearized gauge transformations, lemma 4.2 applies and one concludes that it coincides with the lowest-order homogenous part of a current $k_\mu(x)$, which is a gauge-invariant polynomial in the gauge potential $A^a_\mu(x)$ and its derivatives. This proves that $\check{q}(x)$ is of the form (4.11) and we have thus established the theorem.

7. Proof of the main theorem

We first note that polynomials in the gauge potential $A^a_\mu(x)$ and its derivatives can classified according to their scale dimension, where the potential and each derivative are assigned dimension 1. This corresponds to the scaling behaviour under dilatations

$$A^a_\mu(x) \to \lambda A^a_\mu(\lambda x), \qquad \lambda > 0, \tag{7.1}$$

and it is evident that any given topological field $q(x)$ decomposes into a sum of topological fields with definite scale dimension.

Suppose now that $q(x)$ is a topological field with scale dimension ν and let $\check{q}(x)$ be its lowest-order homogenous part as discussed in sect. 4. The general form of $\check{q}(x)$ is given theorem 4.3, where $c(x)$ and $k_\mu(x)$ may be assumed to have scale dimension ν and $\nu - 1$ respectively. It follows from this that

$$q_1(x) = q(x) - c(x) - \partial_\mu k_\mu(x) \tag{7.2}$$

is a topological field with dimension ν and lowest-order homogenous part $\check{q}_1(x)$ with degree strictly greater than the degree of $\check{q}(x)$.

Proceeding in this way a sequence $q_1(x), q_2(x), \ldots$ of topological fields is thus obtained. The sequence ends when the last field that has been generated vanishes. This has to happen after a finite number of iterations, because the generated fields have the same scale dimension while the degree of their lowest-order homogenous parts is increasing. Once the recursion terminates the sequence of equations can be traced back and it is then obvious that $q(x)$ is equal to a sum of Chern forms plus a divergence term as asserted by theorem 3.1.

Appendix A

A.1 Indices

Throughout these notes the underlying space-time manifold is $\mathbb{R}^n$ and Lorentz indices $\mu, \nu, \ldots$ accordingly run from 1 to n. Since the metric is euclidean it does not matter in which position these indices appear. Multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ are used to label multiple partial derivatives

$$\partial^\alpha = (\partial_1)^{\alpha_1} \ldots (\partial_n)^{\alpha_n}, \qquad \alpha_k \in \{0, 1, 2, \ldots\}, \tag{A.1}$$

with respect to the coordinates $x_1, \ldots, x_n$. For such indices the notations

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha! = \alpha_1! \ldots \alpha_n! \tag{A.2}$$

apply. Vectors in the adjoint representation of the gauge group are labelled by Latin indices $a, b, \ldots$ from the beginning of the alphabet. Repeated indices are always summed over unless stated otherwise. The totally anti-symmetric tensor $\epsilon_{\mu_1 \ldots \mu_n}$ is normalized such that $\epsilon_{12\ldots n} = 1$ and $\delta_{\mu\nu}$ denotes the Kronecker symbol.

A.2 Gauge group

We consider a compact connected Lie group G with Lie algebra $\mathfrak{g}$. With respect to a basis T^a of $\mathfrak{g}$, the structure constants f^{abc} are defined by

$$[T^a, T^b] = f^{abc} T^c. \tag{A.3}$$

The basis may always be chosen such that f^{abc} is real and totally anti-symmetric under permutations of the indices.

The representation space of the adjoint representation of $\mathfrak{g}$ is the Lie algebra itself, i.e. the elements X of $\mathfrak{g}$ are represented by linear transformations

$$\mathrm{Ad}\, X : \mathfrak{g} \mapsto \mathfrak{g}. \tag{A.4}$$

Explicitly $\mathrm{Ad}\, X$ is defined through

$$\mathrm{Ad}\, X(Y) = [X, Y] \quad \text{for all} \quad Y \in \mathfrak{g}. \tag{A.5}$$

The transformation of the basis elements T^a is thus given by

$$\mathrm{Ad}\, X(T^b) = T^a (\mathrm{Ad}\, X)^{ab}, \qquad (\mathrm{Ad}\, X)^{ab} = -f^{abc} X^c, \tag{A.6}$$

where $X = X^a T^a$.

The adjoint representation of the Lie algebra generates a representation of the group G which is also denoted by Ad. A given tensor $t^{a_1 \cdots a_r}$ of rank r is referred to as G-invariant if

$$t^{a_1 \ldots a_r} = (\mathrm{Ad}\, g)^{a_1 b_1} \ldots (\mathrm{Ad}\, g)^{a_r b_r} t^{b_1 \ldots b_r} \tag{A.7}$$

for all group elements g.

A.3 Fields

Gauge potentials are vector fields $A_\mu(x)$, $x \in \mathbb{R}^n$, with values in the Lie algebra $\mathfrak{g}$. In terms of a basis T^a of $\mathfrak{g}$ they are given by

$$A_\mu(x) = A_\mu^a(x) T^a \tag{A.8}$$

with real components $A_\mu^a(x)$. The field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{A.9}$$

is also Lie algebra valued and can thus be expanded in components in the same way as the gauge potential. It is always assumed that the fields which are being considered are infinitely often differentiable with respect to x.

Under an infinitesimal gauge transformation $\omega(x) = \omega^a(x) T^a$, gauge potentials transforms according to

$$\delta A_\mu = D_\mu \omega, \qquad D_\mu = \partial_\mu + \mathrm{Ad}\, A_\mu. \tag{A.10}$$

The transformation law for the field tensor thus reads

$$\delta F_{\mu\nu} = [F_{\mu\nu}, \omega] \tag{A.11}$$

and the same formula applies to the covariant derivatives $D_{\rho_1} \ldots D_{\rho_k} F_{\mu\nu}$.

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