Chiral lattice gauge theories

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Consider a “pure” chiral gauge theory

$$\mathcal{L}(x) = -\frac{1}{2g^2}\,\mathrm{tr}\,\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \overline{\psi}(x)P_+\gamma_\mu D_\mu\psi(x)$$

where the fermions are in an <u>anomaly-free</u> representation

Andrejewski Lectures IV, Berlin, February 2004

What would we like to achieve?

Find a lattice formulation of these theories such that

⋆ locality is preserved

⋆ gauge invariance and the lattice symmetries are unbroken

⋆ the correct continuum limit is obtained to all orders in the gauge coupling

Basic framework

- Quantization through the (euclidean) functional integral

$$\langle \phi_1(x_1) \ldots \phi_k(x_k) \rangle = \frac{1}{\mathcal{Z}} \int_{\mathrm{fields}} \phi_1(x_1) \ldots \phi_k(x_k) \, \mathrm{e}^{-S_{\mathrm{G}} - S_{\mathrm{F}}}$$

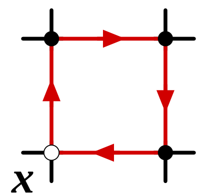
- Space-time $\longrightarrow$ 4d lattice with spacing a

- Lattice Dirac operator D satisfying

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D, \qquad D^\dagger = \gamma_5 D \gamma_5$$

Gauge field $\rightarrow$ link variables $U(x,\mu)$

$$S_{\mathrm{G}} = \frac{1}{g^2}\sum_{\Box} \underset{\substack{\uparrow \\ \text{function of } U_{\Box}}}{S_{\Box}}$$

function of $U_{\Box} =$ 

$= 1 + a^2 F_{\mu\nu}(x) + \mathrm{O}(a^3)$

Choose $S_{\Box}$ such that

- ⋆ all symmetries are preserved
- ⋆ $S_{\Box} = -a^4 \mathrm{tr}\{F_{\mu\nu}F_{\mu\nu}\} + \mathrm{O}(a^5)$
- ⋆ $S_{\Box} < \infty$ implies $\|1 - U_{\Box}\| \le \epsilon \Rightarrow D$ is local, ...

$\psi(x)$, $\overline{\psi}(x)$ = lattice Dirac fields in a representation R of G

$$S_{\mathrm{F}} = a^4 \sum_x \overline{\psi}(x) D \psi(x)$$

Chiral projectors

$$P_{\pm} = \tfrac{1}{2}(1 \pm \gamma_5), \qquad \hat{P}_{\pm} = \tfrac{1}{2}(1 \pm \hat{\gamma}_5), \qquad \hat{\gamma}_5 \equiv \gamma_5(1 - aD)$$

By imposing the constraints

$$\hat{P}_{-}\psi = \psi, \qquad \overline{\psi} P_{+} = \overline{\psi}$$

the right-handed components are eliminated

The chiral constraints are

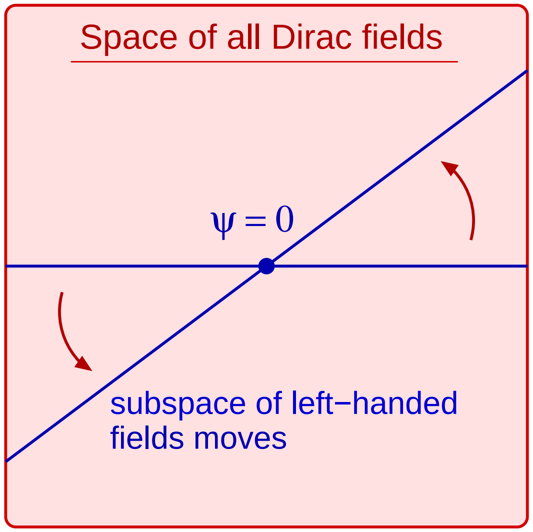
- ⋆ consistent with the field equations
- ⋆ local & gauge-covariant

$\Rightarrow$ completely satisfactory formulation at the classical level

It remains to define the integration measures ...

$$\mathrm{D}[U] = \prod_{x,\mu} \mathrm{d}U(x,\mu) \quad \text{(normalized invariant measure)}$$

Fermion measure



$$\psi(x) = \sum_j c_j v_j(x)$$

$$\uparrow$$

orthonormal basis of left-handed fields

Definition & phase ambiguity

$$\mathrm{D}[\psi] = \prod_j \mathrm{d}c_j \quad \text{(Grassmann integration)}$$

$$v_j(x) \to \sum_l v_l(x) \mathcal{Q}_{lj} \quad \Rightarrow \quad \mathrm{D}[\psi] \to \underbrace{\det \mathcal{Q}}_{\mathrm{e}^{i\phi[U]}} \mathrm{D}[\psi]$$

Similarly define $\mathrm{D}[\overline{\psi}]$ by expanding in some basis functions $\bar{v}_k$

$\Rightarrow$ the fermion integral yields the standard result

$$\int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\,\psi(x_1)\ldots\overline{\psi}(x_l)\,\mathrm{e}^{-S_\mathrm{F}}$$

$$= \mathrm{e}^{-S_\mathrm{eff}} \times \{\text{sum of Wick contractions}\}$$

$$\psi(x)\overline{\psi}(y) = \hat{P}_{-}S(x,y)P_{+}, \qquad DS(x,y) = a^{-4}\delta_{xy}$$

but the partition function

$$\mathrm{e}^{-S_\mathrm{eff}} = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\,\mathrm{e}^{-S_\mathrm{F}} = \det M, \qquad M_{kj} \equiv (\bar{v}_k, Dv_j)$$

has a gauge-field-dependent phase ambiguity

Remarks

- $\mathrm{D}[\psi]$ implicitly depends on the gauge field!

  Fujikawa '79

- The phase of $\mathrm{D}[\psi]$ must be specified to completely define the theory

- Gauge anomaly $\Leftrightarrow$ there is no consistent choice of phase

How can the phase be fixed?

The theory should respect basic principles

* Locality

* Gauge invariance & the lattice symmetries

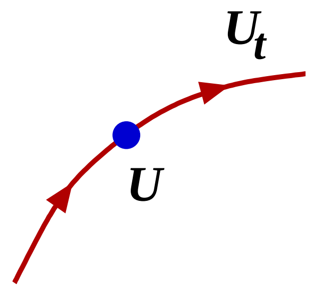
This determines $\mathrm{Im}\, S_{\mathrm{eff}}$ *up to irrelevant local terms*

In the following we are going to work this out a bit ...

Variation of the effective action

Consider a curve of gauge fields

$$U_t(x,\mu) = \mathrm{e}^{ta\eta_\mu(x)} U(x,\mu)$$



Differentiation in direction η at U

$$\delta_\eta S_{\mathrm{eff}} \equiv \left\{ \frac{\mathrm{d}S_{\mathrm{eff}}}{\mathrm{d}t} \right\}_{t=0}$$

$$\to \int \mathrm{d}^4x\, \eta^a_\mu(x) \frac{\delta S_{\mathrm{eff}}}{\delta A^a_\mu(x)} \qquad \text{(in the continuum limit)}$$

Some algebra yields

$$\delta_\eta S_{\rm eff} = -\mathrm{Tr}\{\delta_\eta D\,\hat{P}_- D^{-1} P_+\} + i\underset{\substack{\uparrow \\ \text{"measure term"}}}{\mathfrak{L}_\eta}$$

$$\mathfrak{L}_\eta = i\sum_j (v_j, \delta_\eta v_j) \equiv a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x)$$

The current $j_\mu^a(x)$

- ⋆ characterises the chosen measure
- ⋆ appears on the rhs of the field equations

$\Rightarrow$ $j_\mu^a(x)$ should be a <u>local</u> expression in the gauge field

Gauge transformations

For a gauge variation $\eta_\mu = -\nabla_\mu \omega$ one obtains

$$\delta_\eta S_{\rm eff} = i a^4 \sum_x \omega^a(x) \left\{ [\nabla_\mu^* j_\mu]^a(x) - \mathcal{A}^a(x) \right\}$$

$$\mathcal{A}^a(x) \equiv -\tfrac{1}{2} \,\mathrm{tr} \left\{ R(T^a) \gamma_5 a D(x,x) \right\}$$

$$= c_1 d_R^{abc} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^b(x) F_{\rho\sigma}^c(x) + \mathrm{O}(a) \qquad \text{(\underline{covariant} anomaly)}$$

$$\boxed{\text{Gauge symmetry is preserved} \Leftrightarrow [\nabla_\mu^* j_\mu]^a(x) = \mathcal{A}^a(x)}$$

Integrability condition

To ensure the local integrability of

$$\delta_\eta S_{\text{eff}} = -\text{Tr}\{\delta_\eta D \hat{P}_- D^{-1} P_+\} + i\mathfrak{L}_\eta$$

it is necessary & sufficient that

$$\boxed{\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + a\mathfrak{L}_{[\eta,\zeta]} = i\,\text{Tr}\{\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]\}}$$

Theorem

Any given current $j^a_\mu(x)$ satisfying this equation arises from an underlying fermion measure that is uniquely determined, up to a constant phase factor, on any simply-connected piece of field space

M.L. '99

We are thus left with the task to construct a current $j^a_\mu(x)$ that

- is a local expression in the gauge field
- fulfils the requirement of gauge invariance

$$[\nabla^*_\mu j_\mu]^a(x) = -\tfrac{1}{2}\,\mathrm{tr}\,\{R(T^a)\gamma_5 a D(x,x)\}$$

- satisfies the integrability condition

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + a\mathfrak{L}_{[\eta,\zeta]} = i\,\mathrm{Tr}\{\hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-]\}$$

Note: (1) there is no solution if $d_R^{abc} \neq 0$

(2) $j^a_\mu = \mathrm{O}(a)$ if the representation is anomaly-free

Equivalent cohomology problem

Consider gauge fields on $\mathbb{L}\text{attice} \times \mathbb{R}^2$

$$U(z,\mu),\; A_t(z),\; A_s(z), \qquad\qquad z=(x,t,s)$$

Define

$$q(z)=\tfrac{1}{2}\,\mathrm{Im}\,\mathrm{tr}\left\{\left[\hat{\gamma}_5[D_t\hat{P}_-,D_s\hat{P}_-]+R(F_{ts})\hat{\gamma}_5\right](x,x)\right\}$$

Using the GW relation it can be shown that

$$a^4\sum_x\int \mathrm{d}t\,\mathrm{d}s\,\delta q(z)=0$$

for all local deformations of the gauge field, i.e. q is <u>topological</u>

In the classical continuum limit

$$q(z) = \frac{1}{6} c_1 d_R^{abc} \epsilon_{\mu\nu\rho\sigma\lambda\delta} F^a_{\mu\nu}(z) F^b_{\rho\sigma}(z) F^c_{\lambda\delta}(z) + \mathrm{O}(a)$$

i.e. $q(z)$ has trivial cohomology at $a = 0 \Leftrightarrow d_R^{abc} = 0$

The higher-order terms in the expansion are trivial, because there are no Chern monomials with dimension $d > 6$

Theorem

At any $a > 0$ the following statements are equivalent:

1. $q(z)$ *has trivial cohomology*
2. *there exists a local current $j^a_\mu(x)$ that has all the required properties*

U(1) theory with N left-handed fermions

Anomaly cancellation condition

$$\sum_{\alpha=1}^{N} \mathrm{e}_\alpha^3 = 0$$

$q(z)$ has trivial cohomology if this holds since

$$q = \text{Chern monomial} + \text{divergence term}$$

for any $a > 0$

$\Rightarrow$ abelian chiral gauge theories can be put on the lattice without breaking the gauge symmetry!

M.L. '98, Kikukawa '01

Remarks

- $j_\mu(x)$ is obtained from $q(z)$ by differentiation and one-dimensional integrations

  Kadoh, Kikukawa & Nakayama '03, Kadoh & Kikukawa '04

- The construction of the theory is non-perturbative and completely rigorous

- Global obstructions along non-contractible loops are absent if the e_α are even or paired ($\mathrm{e}_\alpha = \pm\mathrm{e}_{\alpha^*}$)

Non-abelian theories

The cohomology problem has not been solved to date

$\Rightarrow$ no general result is available

Currently the solved cases are

- ⋆ Real & pseudo-real representations (trivial)

- ⋆ $\mathrm{SU}(2) \times \mathrm{U}(1)$ electroweak theory

  Kikukawa & Nakayama '00

- ⋆ All orders of perturbation theory (any gauge group and fermion representation)

  H. Suzuki '00, M.L. '00

In perturbation theory one sets

$$U(x,\mu) = \exp\{gaA_\mu(x)\}$$

and expands in powers of g

$$j^a_\mu(x) = \sum_{k=0}^{\infty} \frac{g^k}{k!} a^{4k} \sum_{x_1,\ldots,x_k}$$

$$\times\, L^{(k)}(x,x_1,\ldots,x_k)^{aa_1\ldots a_k}_{\mu\mu_1\ldots\mu_k} A^{a_1}_{\mu_1}(x_1)\ldots A^{a_k}_{\mu_k}(x_k)$$

Locality:

$L^{(k)}$ decays rapidly if $\|x_j - x\| >$ a few lattice spacings

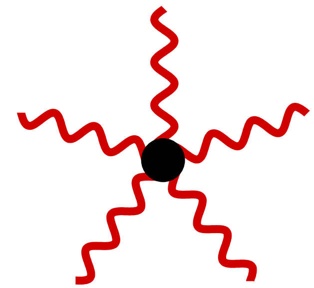
Gauge invariance & integrability:

$=$ set of linear equations for the $L^{(k)}$

Theorem

If $d_R^{abc} = 0$ there exist lattice functions $L^{(k)}$ such that

- ⋆ $j_\mu^a(x)$ *has the required properties to all orders*
- ⋆ $L^{(k)} = 0$ *for* $k \leq 3$
- ⋆ $\mathrm{Im}\, S_{\mathrm{eff}}$ *transforms like a pseudo-scalar*



The $L^{(k)}$ are obtained algebraically through a recursive procedure

Solution is unique up to $S_{\mathrm{eff}} \to S_{\mathrm{eff}} + \Delta S$, where ΔS is local, gauge-invariant, pseudo-scalar & irrelevant

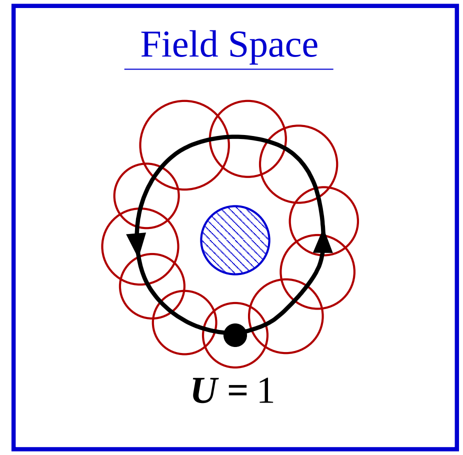
> Anomaly-free chiral gauge theories can be regularized without breaking the gauge symmetry

Further developments

Global anomalies

Example: $G = SU(2),\ R = \frac{\mathbf{1}}{\mathbf{2}}$

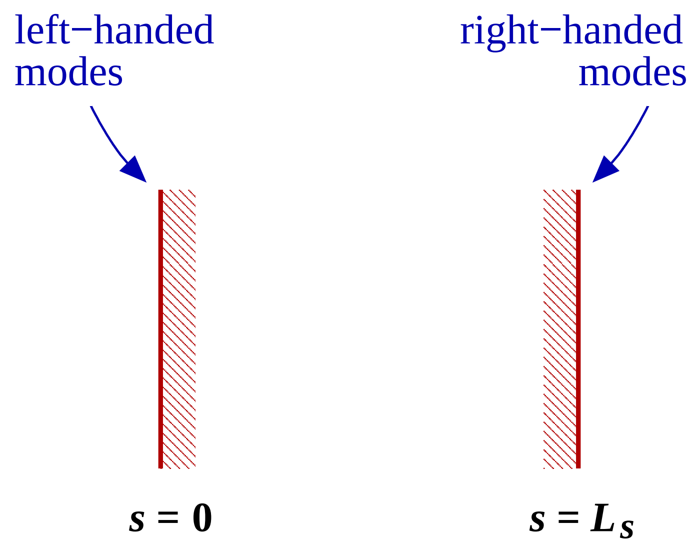
The phase of the fermion measure cannot be fixed globally



Bär & Campos '99ff

4+1 dimensional approach

Using domain-wall instead of GW fermions



Alvarez-Gaumé, S. & V. Della Pietra '85

Kaplan '92, Shamir '93

Aoyama & Kikukawa '99, Kikukawa '01