Wilson quarks and the Banks–Casher relation

Martin Lüscher & Leonardo Giusti, CERN Physics Department

Perspectives and challenges for full QCD lattice calculations

ECT Trento, 5.–9. May 2008

Banks–Casher relation

The chiral condensate in QCD is given by

$$\Sigma = \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{u} u \rangle = \pi \rho(0,0)$$

where $\rho(\lambda, m)$ is the spectral density of the Dirac operator

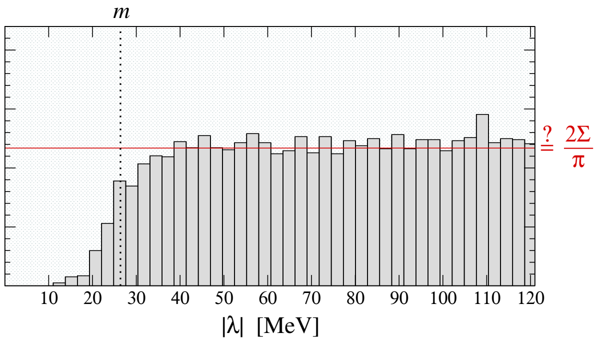
Banks & Casher '80

On the lattice

- the relation remains valid if chiral symmetry is preserved
- may in principle be used to compute the condensate

How about Wilson quarks?

Spectrum of the hermitian Wilson–Dirac operator $\gamma_5 D_m$ on a 48×24^3 lattice



M.L. '07 [JHEP 0707 (2007) 081]

1. Renormalization of spectral observables
2. O(a)-improved Wilson theory
3. ChPT and finite-volume effects
4. Counting low modes on large lattices
5. Numerical studies

Renormalization of spectral observables

First consider the continuum theory

$$D_m\psi = (m + i\lambda)\psi$$

$$D_m^\dagger D_m\psi = \alpha\psi, \qquad \alpha = m^2 + \lambda^2$$

Average no of eigenstates of $D_m^\dagger D_m$ with $\alpha \leq M^2$

$$\nu(M, m) = V \int_{-\Lambda}^{\Lambda} \mathrm{d}\lambda \, \rho(\lambda, m), \qquad M^2 = m^2 + \Lambda^2$$

Is this a renormalizable quantity?

Del Debbio, Giusti, M.L., Petronzio & Tantalo '06

Consider the spectral sums

$$\sigma_k(\mu, m) = \left\langle \mathrm{Tr}\left\{ \left(D_m{}^\dagger D_m + \mu^2 \right)^{-k} \right\} \right\rangle$$

$$= \int_0^\infty \mathrm{d}M \, \nu(M, m) \frac{2kM}{\left(M^2 + \mu^2\right)^{k+1}}$$

- ⋆ Well-defined if $\mu^2 > 0$ and $k \geq 3$
- ⋆ For fixed k, the relation $\nu(M, m) \leftrightarrow \sigma_k(\mu, m)$ is invertible
- ⋆ It is therefore sufficient to understand the renormalization of $\sigma_k(\mu, m)$

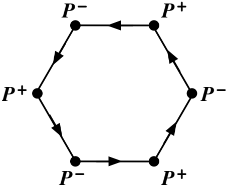
Note that

$$(D_m^{\dagger} D_m + \mu^2)^{-1}$$

is the square of the quark propagator in tmQCD

⇒ introduce N doublets of twisted-mass valence quarks

$$S_{\rm val} = \int \mathrm{d}^4 x \sum_{n=1}^{N} \overline{\psi}_n(x)(D_m + i\mu\gamma_5\tau^3)\psi_n(x)$$



$$P_{ij}^{\pm} = \overline{\psi}_i \gamma_5 \tau^{\pm} \psi_j$$

$$\sigma_3(\mu, m) = -\int \mathrm{d}^4 x_1 \dots \mathrm{d}^4 x_6 \times$$

$$\left\langle P_{12}^{+}(x_1) P_{23}^{-}(x_2) P_{34}^{+}(x_3) P_{45}^{-}(x_4) P_{56}^{+}(x_5) P_{61}^{-}(x_6) \right\rangle$$

The partially quenched theory is renormalized through

$$m_{\mathrm{R}} = Z_{\mathrm{P}}^{-1} m, \qquad \mu_{\mathrm{R}} = Z_{\mathrm{P}}^{-1} \mu, \qquad (P_{ij}^{\pm})_{\mathrm{R}} = Z_{\mathrm{P}} P_{ij}^{\pm}$$

Short-distance singularities

$$P_{ij}^{+}(x) P_{jk}^{-}(y) \underset{x \to y}{\sim} |x-y|^{-3} S_{ik}^{+-}(y)$$

are integrable and the total degree of divergence is negative if $k \geq 3$

Renormalized spectral sums and mode number

$$Z_{\mathrm{P}}^{2k} \sigma_k(Z_{\mathrm{P}}\mu_{\mathrm{R}}, Z_{\mathrm{P}} m_{\mathrm{R}}) = Z_{\mathrm{P}}^{-2} \int_0^{\infty} \mathrm{d}M \, \nu(M, Z_{\mathrm{P}} m_{\mathrm{R}}) \frac{2kM}{\big(\underbrace{Z_{\mathrm{P}}^{-2} M^2}_{M_{\mathrm{R}}^2} + \mu_{\mathrm{R}}^2\big)^{k+1}}$$

$$\nu_{\mathrm{R}}(M_{\mathrm{R}}, m_{\mathrm{R}}) = \nu(Z_{\mathrm{P}} M_{\mathrm{R}}, Z_{\mathrm{P}} m_{\mathrm{R}})$$

O(a)-improved Wilson theory

Consider the hermitian eigenproblem

$$D_m^\dagger D_m \psi = \alpha \psi$$

$$\nu(M, m_\mathrm{q}) = \big\langle \text{No of eigenvectors with } \alpha \leq M^2 \big\rangle, \qquad m_\mathrm{q} = m_0 - m_\mathrm{c}$$

Define spectral sums and introduce twisted-mass valence quarks

$$\sigma_k(\mu, m) = \big\langle \mathrm{Tr}\big\{ \big(D_m^\dagger D_m + \mu^2\big)^{-k} \big\} \big\rangle$$

$$= \int_0^\infty \mathrm{d}M \, \nu(M, m_\mathrm{q}) \frac{2kM}{(M^2 + \mu^2)^{k+1}}$$

$$\sigma_3(\mu, m) = -a^{24} \sum_{x_1,\ldots,x_6} \big\langle P_{12}^+(x_1) P_{23}^-(x_2) \ldots P_{56}^+(x_5) P_{61}^-(x_6) \big\rangle$$

O(a)-improvement & renormalization

$$m_{\rm R} = Z_m(1 + b_m a m_{\rm q}) m_{\rm q} = \frac{Z_{\rm A}(1 + b_{\rm A} a m_{\rm q})}{Z_{\rm P}(1 + b_{\rm P} a m_{\rm q})} m, \quad m\text{: PCAC quark mass}$$

$$\mu_{\rm R} = Z_\mu(1 + b_\mu a m_{\rm q})\mu$$

$$(P^{\pm}_{ij})_{\rm R} = Z_{\rm P}(1 + b_{\rm P} a m_{\rm q}) P^{\pm}_{ij}$$

$\Rightarrow$ renormalized spectral sums

$$\left(Z_{\rm P} \underbrace{\frac{1 + b_{\rm P} a m_{\rm q}}{1 + b_{\rm PP} a m_{\rm q}}}_{\text{short-distance correction}}\right)^{2k} \sigma_k(\mu, m_{\rm q}) \quad \text{where} \quad \mu = \mu(\mu_{\rm R}, m_{\rm R}), \ \ldots$$

Differentiating with respect to μ, it can be shown that

$$Z_\mu Z_{\rm P} = 1, \qquad b_\mu + b_{\rm P} - b_{\rm PP} = 0$$

$$Z_{\rm P} \frac{1 + b_{\rm P} a m_{\rm q}}{1 + b_{\rm PP} a m_{\rm q}} = \frac{1}{Z_\mu (1 + b_\mu a m_{\rm q})} + {\rm O}(a^2)$$

⇒ up to ${\rm O}(a^2)$ terms, the renormalized mode number is given by

$$\nu_{\rm R}(M_{\rm R}, m_{\rm R}) = \nu(M, m_{\rm q}) \quad \text{where} \quad M_{\rm R} = Z_\mu (1 + b_\mu a m_{\rm q}) M,$$

$$m_{\rm R} = Z_m (1 + b_m a m_{\rm q}) m_{\rm q}$$

Note: $b_\mu = -\frac{1}{2} - 0.11 \times g_0^2 + \ldots$ [Frezzotti, Weisz & Sint '01]

Chiral perturbation theory

In the continuum limit and for $V \to \infty$

$$\nu(M,m) =$$

$$\frac{2}{\pi}\Lambda\Sigma V\left\{1 + \frac{M_\pi^2}{32\pi^2 F_\pi^2}\left[3\bar{l}_6 + 3\ln\left(\frac{m_{\rm phys}}{\Lambda}\right) + 2\ln 2 - \frac{\pi}{2}\frac{m}{\Lambda}\right] + \ldots\right\},$$

(where $M^2 = m^2 + \Lambda^2$)

Smilga & Stern '93; Osborn, Toublan & Verbaarschot '99; M.L. & L.G. '08

- The 1-loop correction vanishes when $m \to 0$
- Expected to be fairly small (a few % perhaps) at $M_\pi \leq 300$ MeV, $\Lambda = 50 - 100$ MeV

At small m and moderate Λ

$$\Sigma_{\text{eff}}(M,m) = \frac{\pi}{2}\,\frac{\nu(M,m)}{\Lambda V}$$

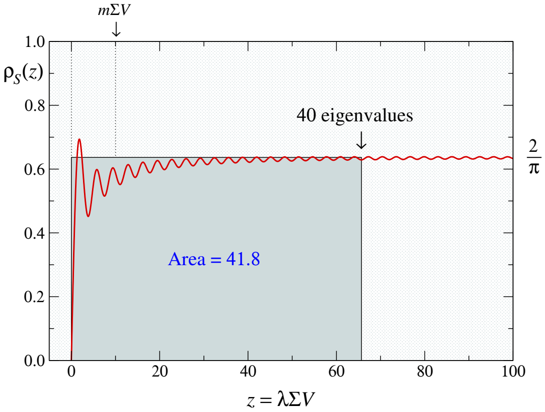
should thus be a good approximation to Σ

NLO ChPT also suggests that the finite-volume effects

$$\frac{\Sigma_{\text{eff}}}{\Sigma_{\text{eff}}\big|_{V=\infty}} - 1 \sim \mathrm{e}^{-\frac{1}{2}M_\Lambda L}, \qquad M_\Lambda^2 = \frac{\Lambda}{m}M_\pi^2$$

are negligible (a fraction of a percent) in the p-regime

When $\Lambda\Sigma V$ is not very large, there could be important threshold effects



$N_f = 2,\ m > 0,\ Q = 0$

Wilke, Guhr & Wettig '97

$\Rightarrow$ Σ_{eff} underestimates Σ by 4.5% in this case

$\Rightarrow$ large lattices and $(p + \epsilon)$-regime ChPT

How to count the low modes

Need a robust method that scales well with the lattice volume

$P_M = \theta(M^2 - D_m^{\dagger} D_m) =$ projector to the low modes

$\mathcal{O}[U] = \mathrm{Tr}\{P_M\}$

$\nu(M, m_{\mathrm{q}}) = \langle \mathcal{O} \rangle$

- ⋆ Relative statistical error scales like $V^{-1/2}$
- ⋆ However, reliably calculating O(V) eigenvalues may not be practical

Stochastic method

$\eta(x)$: gaussian random spinor field, $\langle(\eta,\eta)\rangle = 12V$

$\nu(M, m_{\rm q}) = \langle\widehat{\mathcal{O}}\rangle, \qquad \widehat{\mathcal{O}}[U,\eta] = (\eta, P_M \eta)$

$\mathrm{var}(\widehat{\mathcal{O}}) = \mathrm{var}(\mathcal{O}) + \nu(M, m_{\rm q})$

$\Rightarrow$ the relative error still scales like $V^{-1/2}$

For the computation of

$$P_M \eta = \theta(M^2 - D_m{}^{\dagger} D_m)\eta$$

one may use a rational approximation to the θ-function

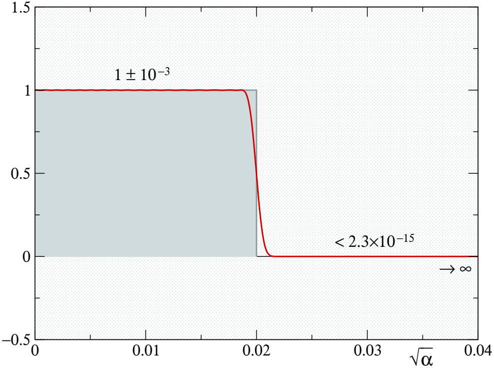
Define:

$$X = \frac{D_m^{\dagger} D_m - M^2}{D_m^{\dagger} D_m + M^2}$$

$$h(X) = \frac{1}{2}\{1 - XP(X^2)\}$$

where $P(X^2)$ = polynomial approximation to $(X^2)^{-1/2}$

$\Rightarrow \; h(X)^4 \simeq \theta(M^2 - D_m^{\dagger} D_m)$



Note:

Shape is independent of V
$\Rightarrow$ total effort $\propto V$

Numerical studies

Using samples of 80 configurations of the CERN–TorVergata ensembles

Del Debbio, Giusti, M.L., Petronzio & Tantalo '07-'08

Lattice parameters

64×32^3 lattice, $\beta = 5.3$, $c_{\rm sw} = 1.90952$, $N_f = 2$

$a = 0.0784(10)$ fm, $L = 2.51(3)$ fm

Renormalization factors

$Z_{\rm A} = 0.75(1)$, $Z_{\rm P}^{-1} = 1.84(3)$ (lattice $\rightarrow \overline{\rm MS}$ at 2 GeV)

Della Morte et al. [ALPHA Collab.] '05

$\Lambda_{\rm R}$	$m_{\rm R}$	$M_{\rm R}$	$\nu_{\rm R}(M_{\rm R}, m_{\rm R})$	$(\Sigma_{\rm eff})^{1/3}$
100	44.1(9)	109.3(4)	75.0(9)(16)	282(4)
	25.3(6)	103.2(1)	68.3(9)(14)	273(4)
	12.4(3)	100.8(1)	65.4(8)(14)	269(4)
70	44.1(9)	82.7(5)	46.8(7)(10)	271(4)
	25.3(6)	74.4(2)	44.1(7)(9)	266(4)
	12.4(3)	71.1(1)	42.3(6)(9)	262(4)

All masses are renormalized in the $\overline{\rm MS}$ scheme at 2 GeV and are given in MeV

May be compared with the $N_f = 2$ JLQCD result

$\Sigma^{1/3} = 251(7)(11)$ MeV   Fukaya et al. '07

extracted from the lowest eigenvalues of the Dirac operator in the ϵ-regime

As usual there are

* finite-volume (including threshold) effects
* higher-order chiral corrections
* lattice-spacing effects

that must be studied and eventually "extrapolated away"

⇒ a larger range of lattices will need to be considered

Conclusions

Spectral projectors provide a new opportunity to study the chiral regime of QCD

- ⋆ Chiral condensate
- ⋆ Ward identities ($\rightarrow Z_{\rm A}, Z_{\rm S}/Z_{\rm P}$)
- ⋆ Topological susceptibility, other low-energy constants, ...

Theoretically clean, moderate effort, small statistical errors, scales favourably

Matching with ChPT may require $(\epsilon + p)$-regime calculations