Renormalization and continuum limit of the gradient flow in non-Abelian gauge theories

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Based in part on work done in collaboration with Peter Weisz

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Definition & basic properties

SU(N) gauge theory, gauge field $A_\mu(x)$

Consider the "flow of fields" $B_\mu(t,x)$, $t \geq 0$, defined by

$$B_\mu|_{t=0} = A_\mu$$

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Atiyah & Bott '82, ... [Morse theory of field space]

Remarks

- *Purely geometric equation*
- *The solution is a gauge-covariant function(al) of* $A_\mu(x)$

- *The evolution in t has a smoothing effect*

$$\underbrace{\partial_t B_\mu = \Delta B_\mu}_{\text{heat equation}} - \partial_\mu \partial_\nu B_\nu + \text{non-linear}$$

$$\Rightarrow \; B_\mu(t,x) = \int \mathrm{d}^4 y\, K_t(x-y) A_\mu(y) + \text{gauge \& non-linear terms}$$

$$K_t(z) = \frac{\mathrm{e}^{-\frac{z^2}{4t}}}{(4\pi t)^2}, \qquad \text{smoothing range} = \boldsymbol{\sqrt{8t}}$$

- *In the quantized theory, a regularization is needed*

- *Gradient flow in lattice gauge theory*

$$V_t(x,\mu)|_{t=0} = U(x,\mu)$$

$$\partial_t V_t(x,\mu) V_t(x,\mu)^{-1} = -a g_0^2 \underbrace{\delta_{x,\mu} S(V_t)}_{\text{gauge force}}$$

$=$ continuous "stout-link smoothing" Morningstar & Peardon '04

- *Observables*

For example

$$E(t,x) = \tfrac{1}{4}\left(G^a_{\mu\nu} G^a_{\mu\nu}\right)(t,x)$$

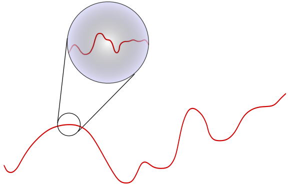
$$\langle E(t_1,x_1)\ldots E(t_n,x_n)\rangle$$

Motivation

⋆ Get insight into the dynamics of non-Abelian gauge fields at different length scales

⋆ Provide a quantum-field theoretical definition of the topological sectors in QCD

Typical gauge-field configurations are nowhere continuous ...



$$\Leftrightarrow \quad \left\langle (F^a_{\mu\nu}\,{}^*F^a_{\mu\nu})(x)(F^b_{\rho\sigma}\,{}^*F^b_{\rho\sigma})(0)\right\rangle \underset{x\to 0}{\propto} (x^2)^{-4}$$

Motivation (cont.)

- ⋆ Non-perturbative renormalization

- ⋆ Multilevel simulation algorithms

Finiteness of the gradient flow

Correlation functions at $t > 0$ do not require renormalization

In QCD, for example,

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha_{\rm s}(q) \left\{ 1 + k_1 \alpha_{\rm s}(q) + \ldots \right\}, \qquad q = (8t)^{-1/2}$$

- Established to all orders of perturbation theory
- Confirmed through numerical simulations

Proof of finiteness (ML & Weisz '10)

Flow equation = Langevin equation w/o noise

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \eta_\mu$$

$$\langle \eta_\mu(t,x)\eta_\nu(s,y)\rangle = 2g_0^2\delta_{\mu\nu}\delta(t-s)\delta(x-y)$$

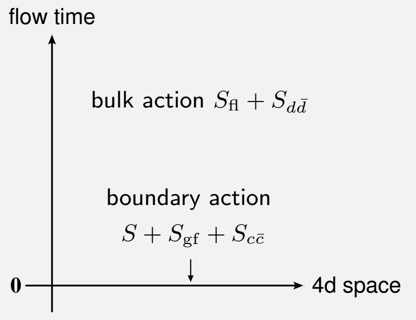
Basic idea: n-point correlation functions

$$\langle B_{\mu_1}(t_1,x_1)\ldots B_{\mu_n}(t_n,x_n)\rangle$$

are those of a local field theory in a 5d half-space!

$\Rightarrow$ Feynman diagrams, power-counting, local counterterms

Zinn-Justin & Zwanziger '88



$$S_{\text{fl}} = \int_0^\infty \mathrm{d}t \int \mathrm{d}^4x \, \underset{\substack{\uparrow \\ \text{Lagrange multiplier}}}{L_\mu^a} \left(\partial_t B_\mu^a - D_\nu G_{\nu\mu}^a - \underset{\substack{\uparrow \\ \text{gauge mode damping}}}{\alpha_0 D_\mu \partial_\nu B_\nu^a}\right)$$

5d theory has an exact BRS symmetry!

Bulk counterterms?

Far away from the boundary $t = 0$ the propagators are

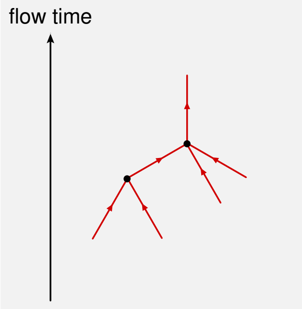
$$\langle \widetilde{B}_\mu(t,p)\widetilde{L}_\nu(s,-p)\rangle \propto \theta(t-s)\left\{\delta_{\mu\nu}\mathrm{e}^{-(t-s)p^2} + \text{gauge terms}\right\}$$

$$\langle BB\rangle = \langle LL\rangle = 0$$

and there are only LB^2 and LB^3 vertices

$\Rightarrow$ no loop diagrams

$\Rightarrow$ no counterterms needed



Boundary counterterms?

- 4d counterterms depending on $A_\mu, c, \bar{c}$ only

- Power counting shows that

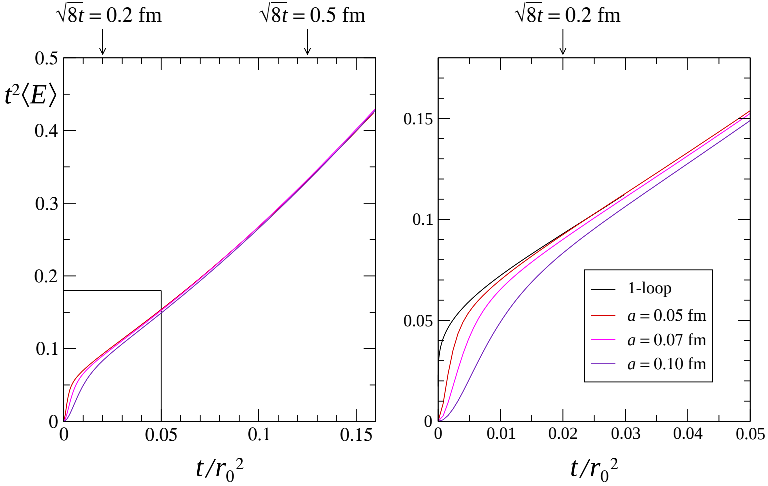
$$\int_{t=0} \mathrm{d}^4x \left\{ z L_\mu^a A_\mu^a + \text{ghost term} \right\}$$

  is the only possible additional counterterm, but this term is excluded by the BRS symmetry

$\Rightarrow$ **Theory is finite after the usual 4d renormalization!**

Finiteness in lattice gauge theory

On the lattice, finiteness holds at flow times t given in physical units



However . . .

- *At fixed a, the flow eventually drives any field to a local minimum of the action*

- *Since the minimizing fields depend on the lattice action, so do the results obtained in this limit*

- *The lattice-spacing effects are therefore expected to grow as $t \to \infty$*

Illustration (I): U(1) theory in 3d

For any value of the coupling $g_0 > 0$, the string tension

$$\sigma \geq \text{const} \times \exp\left\{-\frac{k}{ag_0^2}\right\}$$

does not vanish! Göpfert & Mack '82

$\Rightarrow$ Continuum limit is reached non-uniformly in r

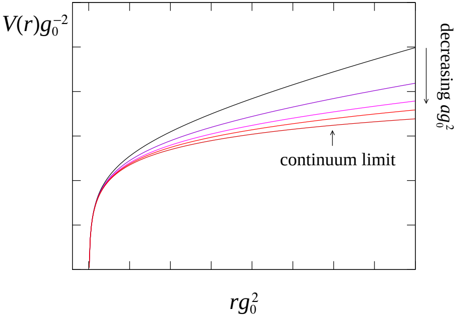


Illustration (II): Twisted-mass QCD

Isospin is broken by lattice effects

$$m_{\pi^+} - m_{\pi^0} = \mathrm{O}(a^2) > 0$$

$$\frac{\sum_{\vec{x}} \langle \pi^0(x) \pi^0(0) \rangle}{\sum_{\vec{x}} \langle \pi^+(x) \pi^+(0) \rangle} \propto \mathrm{e}^{x_0 \Delta m_\pi}$$

Gets arbitrarily large as $x_0 \to \infty$ at fixed a

$\Rightarrow$ *Infrared-enhanced lattice effects are commonplace!*

$\Rightarrow$ *In the case of the flow, universality as $a \to 0$ is guaranteed only when t is held fixed in physical units*

Application: Definition of the topological charge

Gauge-covariant frequency splitting

$$A_\mu(x) = B_\mu(t,x) + \underbrace{\xi_\mu(t,x)}_{\uparrow \text{ “quantum fluctuation”}}$$

Suggests to define

$$Q(t) = \int \mathrm{d}^4x \, q(t,x), \qquad q(t,x) = \frac{1}{32\pi^2} \left(G^a_{\mu\nu} {}^*G^a_{\mu\nu} \right)(t,x)$$

For any $t > 0$, the cumulants

$$C_n(t) = \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_{n-1} \langle q(t,x_1) \ldots q(t,x_n) \rangle_{\mathrm{c}}$$

are well defined and do not require renormalization

Moreover, since

$$\partial_t q = \frac{1}{8\pi^2} \partial_\mu \{ \underbrace{\partial_t B_\nu^{a*} G_{\mu\nu}^a}_{\text{gauge invariant}} \}$$

partial integration shows that C_n **is independent of** t**!**

On the lattice, and **at fixed t/r_0^2**,

$$C_n(t) = a^{4n-4} \sum_{x_1,\ldots,x_{n-1}} \langle \hat{q}(t,x_1)\ldots\hat{q}(t,x_n)\rangle_{\mathrm{c}} + \mathrm{O}(a^2)$$

$\uparrow$

any reasonable lattice expression for $q(t,x)$

In the SU(3) theory, the lattice effects at $\sqrt{8t} \simeq r_0$ are small and

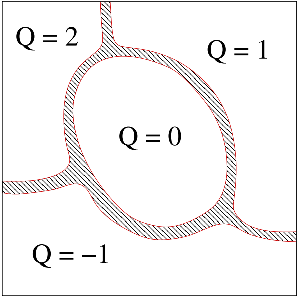
$$\lim_{a\to 0} C_2(t)^{1/4} = 187.4(3.9)\,\mathrm{MeV}$$

Agrees with computations using GW fermions or the "universal formula"

Del Debbio et al. '05, Giusti et al. '02,'04, ML '04, Giusti & ML '09, ML & Palombi '10

Actually, as $a \longrightarrow 0$, the topological sectors emerge dynamically

suppressed like $\sim a^6 \;\longrightarrow$



field space

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Conclusions

The gradient flow allows non-Abelian gauge theories to be probed at different length scales

- Conceptually well-founded
- Technically attractive

Many applications, most of them still to be explored

- Topological sectors
- Non-perturbative renormalization
- Simulation algorithms with improved scaling properties