31st International Symposium on Lattice Field Theory

Mainz, Germany, July 29 – August 03 2013

Future applications of the Yang–Mills gradient flow in lattice QCD

Martin Lüscher, CERN Physics Department

- Flow equations & observables
- Chiral condensate
- Small flow-time expansion of local fields
- Wilson's RG revisited

Thanks to
Agostino Patella, Roberto Petronzio, Stefan Schaefer, Hiroshi Suzuki and Peter Weisz

Yang–Mills gradient flow

Flow of gauge potentials $B_\mu(t,x)$, $t \geq 0$, defined by

$$B_\mu|_{t=0} = A_\mu$$

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Atiyah & Bott '82, ... [Morse theory of field space]

Associated flow $\chi(t,x)$ of quark fields

$$\chi|_{t=0} = \psi$$

$$\partial_t \chi = \Delta\chi, \qquad \Delta = \not{D}^2 \text{ or simply } \Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu$$

- *Smoothing property*

$$B_\mu(t,x) = \int \mathrm{d}^4y\, K_t(x-y)A_\mu(y) + \text{gauge \& non-linear terms}$$

$$\chi(t,x) = \int \mathrm{d}^4y\, K_t(x-y)\psi(y) + \ldots$$

$$K_t(z) \propto \exp\Big\{-\frac{z^2}{4t}\Big\}, \qquad \text{smoothing range} = \boldsymbol{\sqrt{8t}}$$

- *In the quantized theory, a regularization is needed*

Lattice, dimensional regularization

- *Observables*

  Gauge-invariant composite fields

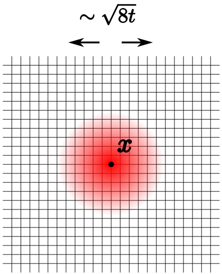
  $$E_t = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

  $$P_t^{rs} = \overline{\chi}_r \gamma_5 \chi_s, \qquad S_t^{rs} = \overline{\chi}_r \chi_s$$

  Consider correlation functions of these

  Similar to "stout" link and source smearing

  Morningstar & Peardon '04; Güsken '90; Alexandrou et al. '91



- *Renormalization*

  $$\mathcal{O}_{\mathrm{R},t} = (Z_\chi)^{\frac{1}{2}(n+\bar{n})} \mathcal{O}_t, \qquad n, \bar{n} = \text{ no. of } \chi \text{ and } \overline{\chi} \text{ fields in } \mathcal{O}$$

  Note: Z_χ is independent of t

  Zinn–Justin & Zwanziger '88; ML & Weisz '11; ML '13

Chiral condensate

Under $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ the chiral densities

$$S_t^{rs} \pm P_t^{rs}, \qquad r, s \in \{u, d\}$$

transform according to the $(\frac{1}{2}, \frac{1}{2})$ representation

$\Rightarrow$ the "time-dependent condensate"

$$\Sigma_t = -\langle S_t^{uu} \rangle$$

is an order parameter for spontaneous chiral symmetry breaking!

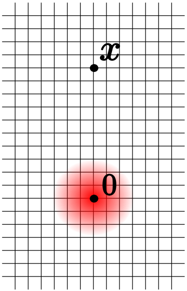
⋆ *Requires only multiplicative renormalization*

⋆ *Accurately calculable on the lattice*

Relation to Σ

The PCAC relation implies

$$\Sigma_t = -\frac{M_\pi^2 F_\pi}{2G_\pi}\int \mathrm{d}^4x\, \underbrace{\langle P^{ud}(x)P_t^{du}(0)\rangle}_{\underset{\vec{p}=0,\, x_0\to\infty}{\sim}\; -\frac{G_\pi G_{\pi,t}}{M_\pi}\mathrm{e}^{-M_\pi x_0}}$$

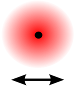


In the chiral limit, the pion pole dominates

$$\Rightarrow \quad \Sigma = \lim_{m_u, m_d\to 0} \Sigma_t \frac{G_\pi}{G_{\pi,t}}$$

Chiral perturbation theory

$S_t^{uu}, P_t^{ud}, \ldots$ are represented by local fields



$\ll M_\pi^{-1}$

$$\Rightarrow \quad \Sigma_t \frac{G_\pi}{G_{\pi,t}} = \Sigma \left\{ 1 - \frac{3M_\pi^2}{32\pi^2 F_\pi^2} \ln(M_\pi^2/\Lambda_t^2) + \ldots \right\}$$

$\bar{l}_t = \ln(\Lambda_t^2/M^2)\big|_{M=140\,\mathrm{MeV}}$: New (time-dependent) LEC

Illustration

2+1 flavours, O(a) improved

64×32^3, $a = 0.090$ fm

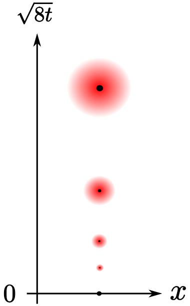
$M_\pi = 203$ MeV, $M_K = 520$ MeV

PACS-CS '10; ML & Schaefer '12

$\sqrt{8t}$ [fm]	$a^3 \Sigma_t \dfrac{Z_P^{-1} G_\pi}{G_{\pi,t}}$
0.4	0.003962(61)
0.5	0.003872(55)
0.6	0.003785(51)
0.7	0.003711(48)

$$\Rightarrow \; \Sigma_t \frac{G_\pi}{G_{\pi,t}} = [287(2)\,\mathrm{MeV}]^3 \;\; @ \; \sqrt{8t} = 0.5\,\mathrm{fm} \;\; (\overline{\mathrm{MS}} \text{ at 2 GeV})$$

Small flow-time expansion



General form of the expansion

$$\mathcal{O}_t(x) \underset{t\to 0}{\sim} \sum_k c_k(t)\phi_k(x)$$

$\phi_k(x)$: renormalized local fields at $t = 0$

The asymptotic behaviour of the coefficients

$$c_k(t) \underset{t\to 0}{\propto} t^{\frac{1}{2}(d_k - d_{\mathcal{O}})}\bar{g}^{\nu_k}\{1 + \mathrm{O}(\bar{g}^2)\}, \qquad \bar{g} \text{ at scale } (8t)^{-1/2}$$

is determined by the renormalization group

ML & Weisz '11

Example: scalar densities

Use RGI normalization for the quark mass matrix M and all fields

$$\Rightarrow \quad S_t^{rs}(x) = c_0(t)M^{rs} + c_1(t)\mathrm{tr}\{M^2\}M^{rs} + c_2(t)(M^3)^{rs} + c_3(t)S^{rs}(x) + \mathrm{O}(t)$$

$$c_0(t) = -\frac{3}{8\pi^2 t}\{1 + \mathrm{O}(\bar{g}^2)\}$$

$$c_3(t) = (2b_0\bar{g}^2)^{-8/9}\{1 + \mathrm{O}(\bar{g}^2)\} = \frac{G_{\pi,t}}{G_\pi} + \mathrm{O}(t)$$

$\Rightarrow \quad \Sigma \simeq \Sigma_t \dfrac{G_\pi}{G_{\pi,t}}$ if t and M are such that $c_0(t)M^{uu}/c_3(t) \ll \Sigma$

A broader perspective

Represent gauge-invariant local fields through fields at positive flow time

$$\text{“}\phi(x) = c(t)\mathcal{O}_t(x) + \mathrm{O}(t)\text{”}$$

For example

- ⋆ Energy-momentum tensor   H. Suzuki '13
- ⋆ Effective electro-weak Hamiltonian

Big plus: *Renormalization & O(a)-improvement are radically simplified*

$$\chi_r(t,x) \to \{Z_\chi(1 + b_\chi a m_{\mathrm{q},r})\}^{1/2}\chi_r(t,x)$$ ML '13

$$\chi(t,x)\overline{\chi}(s,y) \to a^8 \sum_{v,w} K(t,x;0,v)\{S(v,w) - \boldsymbol{a c_{\mathrm{fl}}}\,\delta(v-w)\}K(s,y;0,w)^\dagger$$

However . . .

- Coefficients must be accurately calculated
  - ◇ Perturbation theory — H. Suzuki '13
  - ◇ Using Ward identities — ML '13; Del Debbio, Patella & Rago '13
  - ◇ Step scaling?
- Need a scaling "window"

  $a \ll \sqrt{8t} \ll$ relevant low-energy scales

Wilson's RG revisited

Use the "blockspin" RG for

- ⋆ *non-perturbative renormalization*
- ⋆ *the construction of coarse-grid actions*

Wilson '79

An assumption implicitly made at the time was that expectation values of iteratively blocked Wilson loops have a continuum limit

We may now

- ⋆ *replace the blocking by the gradient flow*
- ⋆ *and use step scaling*

Step scaling (massless theory)

ML, Weisz & Wolff '91

Consider a gauge coupling and fields that run with the lattice size L

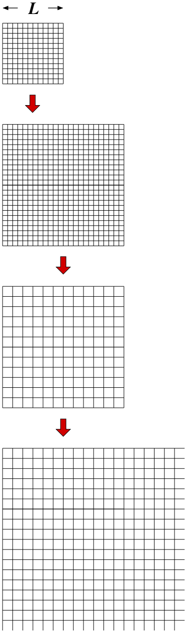
Using the gradient flow, a possible choice is

$$\bar{g}^2 = \text{constant} \times t^2 \langle E_t \rangle \big|_{\sqrt{8t}=\frac{1}{3}L}$$

ML '10; Fodor et al. '12; Fritzsch & Ramos '13

Usually take $a \to 0$ in the evolution step

Solves the non-perturbative renormalization problem **in the continuum theory**



Construction of improved actions

At fixed $\bar{g}$, renormalized quantities are constant up to lattice effects

$\Rightarrow$ *tune action & fields so as to minimize the effects*

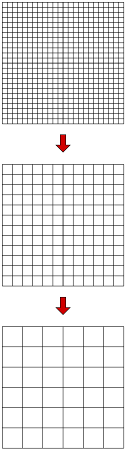
May now be technically feasible by matching

$$\langle \mathrm{tr}\{G_{\mu\nu}G_{\mu\nu}\}\rangle, \ \langle\overline{\chi}\chi\rangle, \ \langle\overline{\chi}\sigma_{\mu\nu}G_{\mu\nu}\chi\rangle,$$

$$\langle(\overline{\chi}\,\Gamma\chi)(\overline{\chi}\,\Gamma\chi)\rangle, \quad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu},$$

etc.

in a range of flow time t



Concluding remarks

Many ideas presented here are still in their infancy

However, the Yang–Mills gradient flow and its extension to the quark fields stand on solid theoretical ground

And several applications are already in a ready-to-use state!

Related talks & posters by

Mattia Dalla Brida

Nathan Brown

Mattia Bruno

Patrick Fritzsch

Kostas Orginos

Jarno Rantaharju

Agostino Patella

Gregory Petropoulos

Alberto Ramos

Felix Stollenwerk