How to derive the Virasoro algebra from dilatation invariance

(talk given at MPI Munich, December 1988)

1. Historical remark

What I am going to say is essentially based on an unpublished manuscript written in 1976 [1]. At that time conformal invariance in quantum field theory became an unpopular subject, because gauge theories turned out to be asymptotically free at high energies so that the scaling of cross-sections would be governed by logarithmically modified canonical power laws rather than anomalous powers of the energy scale. Many of the things discussed in recent years in two-dimensional conformal theories have been known to experts in those days (for a review and a long list of references see [2]).

2. Goal of the lecture

From the point of view of statistical mechanics, the assumption of conformal invariance at the critical point (which one usually makes) is not very plausible, at least compared to dilatation invariance which follows naturally from renormalization group ideas. I here would like to show that in 2-dimensional models dilatation invariance plus the existence of an energy momentum tensor $\theta_{\mu\nu}(x)$ (not necessarily traceless) implies general conformal invariance on the level of the field algebra (the invariance of the ground state $|0\rangle$ is a separate question). In particular, without any further assumptions, the correlation functions of $\theta_{\mu\nu}(x)$ are exactly calculable. All these results will be established in a rigorous axiomatic framework.

3. List of assumptions

(a) Wightman axioms (except asymptotic completeness)

- Hilbert space $\mathcal{H}$ of physical states
- Unitary representation $U(\Lambda, a)$ of Poincaré group
- $P_0 \geqslant 0$, $P^2 \geqslant 0$, groundstate $|0\rangle$ unique
- local fields $\phi(x)$, $\Psi(y)$, ... obeying Fermi- or Bose-statistics, depending on "spin".

(b) Dilatation symmetry

There are unitary operators $V(\lambda)$, $\lambda > 0$, such that

$$V(\lambda)|0\rangle = |0\rangle,$$

$$V(\lambda)\,\phi(x)\,V(\lambda)^{-1} = \lambda^d\,\phi(\lambda x) \quad \text{for the basic fields } \phi(x).$$

d is the "dimension" of ϕ.

(c) Energy-momentum tensor $\theta_{\mu\nu}$

$\theta_{\mu\nu}$ is a basic field of dimension 2 with

$$\theta_{\mu\nu} = \theta_{\nu\mu}, \quad \theta_{\mu\nu}{}^{+} = \theta_{\mu\nu}, \qquad \partial^\mu \theta_{\mu\nu} = 0.$$

I shall also assume that $\theta_{\mu\nu}$ generates the translations, i.e.

$$\int dx^1 \, [\theta_{0\mu}(x), \phi(y)] = -i \, \partial_\mu \phi(y)$$

for all basic fields ϕ.

This is all what is needed; the results that follow are rigorously deduced from (a) - (c).

4. $\theta_{\mu\nu}$ is traceless

To prove this statement it is convenient to introduce light cone coordinates:

$$x_+ = x^0 + x^1, \qquad \partial_+ = \frac{1}{2}(\partial_0 + \partial_1),$$

$$x_- = x^0 - x^1, \qquad \partial_- = \frac{1}{2}(\partial_0 - \partial_1)$$

$$\theta_{++} = \frac{1}{2}(\theta_{00} + \theta_{11} + 2\theta_{01})$$

$$\theta_{--} = \frac{1}{2}(\theta_{00} + \theta_{11} - 2\theta_{01})$$

$$\theta_{+-} = \theta_{-+} = \frac{1}{2}(\theta_{00} - \theta_{11})$$

Then we have

$$\partial_- \theta_{++} + \partial_+ \theta_{-+} = 0,$$

$$\partial_- \theta_{+-} + \partial_+ \theta_{--} = 0.$$

Lorentz - boosts:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \mathrm{ch}\,\chi & \mathrm{sh}\,\chi \\ \mathrm{sh}\,\chi & \mathrm{ch}\,\chi \end{pmatrix}$$

(1) $$U(\Lambda)\,\theta_{++}(x)\,U(\Lambda)^{-1} = e^{2\chi}\,\theta_{++}(e^{\chi}x_+, e^{-\chi}x_-).$$

Dilatations:

(2) $$V(\lambda)\,\theta_{++}(x)\,V(\lambda)^{-1} = \lambda^2\,\theta_{++}(\lambda x)$$

Combination of (1) and (2) with $\lambda = e^{-\chi}$:

$$U(\Lambda)V(\lambda)\,\theta_{++}(x)\,V(\lambda)^{-1}U(\Lambda)^{-1} = \theta_{++}(x_+, \lambda^2 x_-)$$

It follows that

$$\langle 0|\,\theta_{++}(x)\,\theta_{++}(y)\,|0\rangle = A\,(x_+ - y_+ - i\varepsilon)^{-4}.$$

Thus, by positivity

$$\partial_-\,\theta_{++}(x)\,|0\rangle = 0,$$

and therefore, by the Reeh - Schlieder theorem,

(3) $$\partial_-\,\theta_{++} = 0.$$

Similarly, one proves that $\partial_+\,\theta_{--} = 0$.

Finally, using energy-momentum conservation, we have

$$\partial_+ \theta_{+-} = \partial_- \theta_{+-} = 0 \quad \Rightarrow \quad \theta_{+-} = \text{constant},$$

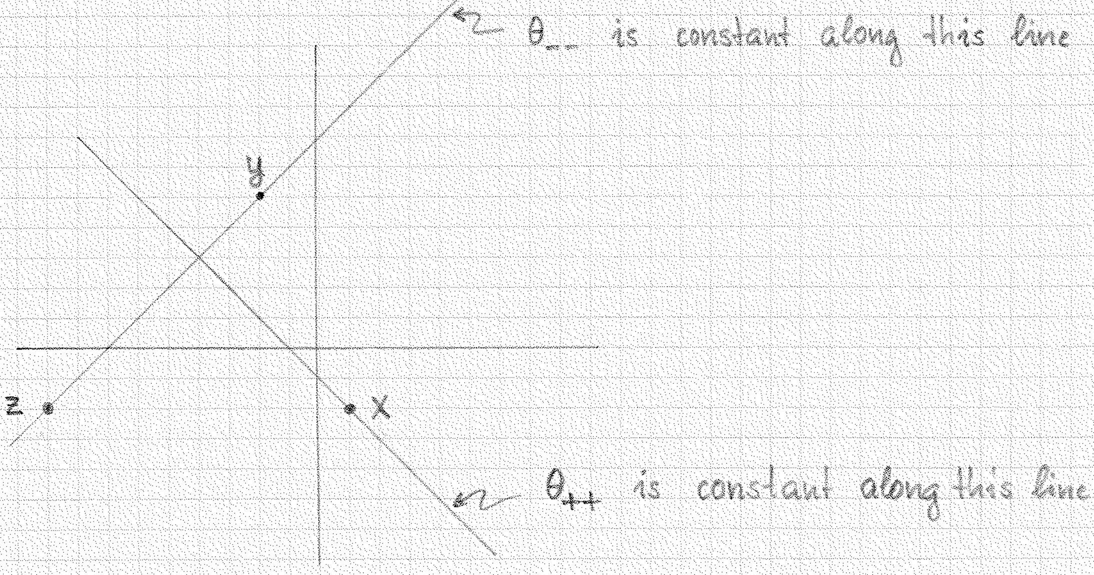
and since θ_{+-} is a field of dimension 2, it follows that $\theta_{+-} = 0$. In other words, $\theta^\mu{}_\mu = 0$.

5. Commutators

From (3) it follows that θ_{++} depends only on x_+, and similarly θ_{--} is a function of x_- only. From locality, we thus conclude

$$[\theta_{++}(x), \theta_{--}(y)] = 0 \qquad \text{for all } x, y$$

(cf. figure). In what follows, I shall discuss the commutator of θ_{++} with itself using the simplified notation $\theta_{++} \to \theta$, $x_+ \to x \in \mathbb{R}$.



<u>Theorem:</u> There exists a number $c \geq 0$ such that

(4) $$[\theta(x), \theta(y)] = \frac{c}{6\pi} i^3 \delta'''(x-y) + 4i\,\delta'(x-y)\,\theta(y) - 2i\,\delta(x-y)\,\partial_y \theta(y)$$

c is called the "central charge".

<u>Proof:</u> By locality, $[\theta(x), \theta(y)] = 0$ for $x \neq y$. Define

$$O_k(x) = \frac{i}{k!} \int dz\; z^k \, [\theta(x+z), \theta(x)], \qquad k = 0, 1, 2, \ldots$$

O_k is a local hermitian field of dimension $d = 3 - k$.
Thus,

$$\langle 0 | O_k(x)\, O_k(y) | 0 \rangle = A_k \, (x - y - i\varepsilon)^{2k-6}.$$

$$\overset{k \geq 3}{=} (-1)^{k-3} A_k \int \frac{dp}{2\pi}\, e^{-ip(x-y)}\, \delta^{(2k-6)}(p)$$

This distribution is not positive for $k > 3$ and hence

$$O_k = 0 \qquad \text{for} \quad k > 3.$$

Furthermore, $O_3(x)$ is independent of x. By locality, O_3 commutes with all basic fields. It is therefore proportional to the unit operator:

$$O_3 = -\frac{c}{6\pi}.$$

Next, recall that $\theta(x)$ generates translations:

$$\int dx\; [\theta(x), \theta(y)] = -2i\, \partial_y \theta(y).$$

Thus,

$$O_0(x) = 2\partial_x \theta(x).$$

Suppose now $|\Psi\rangle \in \mathcal{H}$ is arbitrary. Then,

$$\langle\Psi|\,[\theta(x+z),\theta(x)]\,|0\rangle = \sum_{k=0}^{K} \delta^{(k)}(z)\,\Psi_k(x),$$

where $K<\infty$, and $\Psi_k(x)$ are some distributions. Obviously,

$$\Psi_k(x) = -i(-1)^k \langle\Psi|\,O_k(x)\,|0\rangle.$$

In particular, $\Psi_k = 0$ for $k>3$ and it follows that

$$[\theta(x+z),\theta(x)] = -i\sum_{k=0}^{3} (-1)^k\,\delta^{(k)}(z)\,O_k(x)$$

holds on the vacuum and hence as an operator identity by the usual Reeh-Schlieder argument.

Summarizing, we have shown that

$$[\theta(x),\theta(y)] = \frac{c}{6\pi}\, i^3\,\delta'''(x-y) + i\delta'(x-y)\,O_1(y) - 2i\,\delta(x-y)\,\partial_y\theta(y).$$

To determine O_1 we use

$$[\theta(x),\theta(y)] = -[\theta(y),\theta(x)]$$

to show that $\partial_x O_1(x) = 4\,\partial_x\theta(x)$, which implies $O_1 = 4\theta$ by locality and dilatation invariance. Thus, we have completely proved the commutation rule (4).

Finally, we have to show that $c \geq 0$. From the commutation rule it follows that

$$\langle 0 | [\theta(x), \theta(y)] | 0 \rangle = \frac{c}{6\pi} i^3 \delta'''(x-y),$$

because $\langle 0 | \theta | 0 \rangle = 0$ by translation and dilatation invariance. On the other hand, we have shown that

$$\langle 0 | \theta(x) \theta(y) | 0 \rangle = A (x-y-i\varepsilon)^{-4} \tag{5}$$

which, using

$$\delta'''(x) = -\frac{6}{2\pi i} \{ (x-i\varepsilon)^{-4} - (x+i\varepsilon)^{-4} \},$$

implies

$$A = \frac{c}{2\pi^2}.$$

The Fourier representation of (5) reads

$$\langle 0 | \theta(x) \theta(y) | 0 \rangle = \frac{c}{12\pi^2} \int_0^\infty dp \, p^3 \, e^{-ip(x-y)}$$

so that $c \geq 0$ is necessary to insure positivity of the 2-point function. ■

6. Interpretation

For any smooth rapidly decaying test-function $f(x)$ define

$$\theta(f) = \int dx \, f(x) \theta(x).$$

Then, we have

$$[\theta(f), \theta(g)] = i^3 \frac{c}{12\pi} \int dx\, (f g''' - g f''') + 2i\, \theta([f,g]),$$

$$[f,g](x) = f(x)\, g'(x) - g(x)\, f'(x).$$

This the Lie bracket associated with the group of diffeomorphisms of $\mathbb{R}$. To see this, note that

$$x \to x + \varepsilon f(x) \quad , \quad \varepsilon \text{ infinitesimal},$$

defines an infinitesimal diffeomorphism. Defining the associated variation δ_f through

$$\delta_f F(x) = f(x)\, F'(x)$$

we have

$$[\delta_f, \delta_g] = \delta_{[f,g]}.$$

Thus, $\theta(f)$ is a representation of a central extension of the Lie algebra of the group of diffeomorphisms of $\mathbb{R}$. This algebra is therefore a symmetry of the system in the sense that it is represented by local hermitian automorphisms of the algebra of fields.

7. n-point functions of $\theta(x)$

The algebra (4) completly determines the vacuum expectation values

$$w_n(x_1,\dots,x_n) = \langle 0|\, \theta(x_1)\dots\theta(x_n)\,|0\rangle .$$

To see this, define

$$\theta^{(-)}(x) = \int_0^\infty \frac{dp}{2\pi}\, e^{-ipx}\, \tilde{\theta}(p)$$

$$\theta^{(+)}(x) = \theta(x) - \theta^{(-)}(x)$$

where

$$\tilde{\theta}(p) = \int dx\; e^{ipx}\,\theta(x) .$$

By the spectrum condition, we have

$$\theta^{(-)}(x)\,|0\rangle = 0 \;, \qquad \langle 0|\,\theta^{(+)}(x) = 0 ,$$

and (4) implies

$$[\theta^{(-)}(x), \theta(y)] = \frac{c}{2\pi^2}(x-y-i\varepsilon)^{-4} - \frac{2}{\pi}(x-y-i\varepsilon)^{-2}\,\theta(y)$$
$$- \frac{1}{\pi}(x-y-i\varepsilon)^{-1}\,\partial_y\theta(y) .$$

Thus, we have

$$w_n = \langle 0|\,[\theta^{(-)}(x_1), \theta(x_2)\dots\theta(x_n)]\,|0\rangle$$

and working out the commutator yields w_n in terms of

w_{n-1} and w_{n-2}. Continuing in this way, one obtains w_n in the form of a sum of products of

$$(x_i - x_j - i\varepsilon)^{-k}, \qquad i<j, \qquad k = 1,2,\dots,5.$$

In particular, w_n is a boundary value of a meromorphic function $\hat{w}_n(z_1,\dots,z_n)$, $z_k \in \mathbb{C}$, which is totally symmetric and which is regular for $z_i \neq z_k$.

8. Conformal invariance

Suppose O is any polynomial of local fields smeared with test functions of compact support. Then,

$$[\theta(x), O] = 0 \qquad \text{for large } x$$

and we may define the conformal variations

$$\delta_k O = \frac{1}{2}\int dx\; x^k\, [\theta(x), O], \qquad k = 0,1,2.$$

According to the discussion in §6, these are just the infinitesimal changes of O under the diffeomorphisms

$$x \to x + \varepsilon x^k.$$

It is easy to convince one-self that these transformations generate the group $Sl(2,\mathbb{R})$, which acts on x through

$$x \to \frac{ax+b}{cx+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{R})$$

(recall that x is really x_+; there is another $Sl(2,\mathbb{R})$ acting on x_-). This group is referred to as the (restricted) conformal group.

The question now is whether there are operators J_k such that

$$J_k^{+} = J_k\,, \qquad J_k|0\rangle = 0, \qquad \delta_k O = [J_k, O].$$

If true, one would conclude that

$$\langle 0|\delta_k O|0\rangle = 0,$$

i.e. $\langle 0|O|0\rangle$ satisfies the conformal Ward identities ("infinitesimal or weak conformal invariance").
It is possible to prove that J_0 and J_1 always exist; in fact J_0 generates the translations. J_1 generates dilatations, but one should not conclude that J_1 and the generator of $V(\lambda)$ are the same (the associated dimensions of the fields can be different!). The generator J_2 of the special conformal transformations does, however, not always exist. A counterexample is given in the appendix.

The correlation functions of $\theta(x)$ are, however, always conformally invariant:

$$(6) \qquad \sum_{j=1}^{n} \langle 0|\theta(x_1)\ldots\delta_k\theta(x_j)\ldots\theta(x_n)|0\rangle = 0 \qquad (k=0,1,2).$$

To prove this relation, first note that

$$\delta_k\theta(x) = -i\,D_k\theta(x), \qquad D_k = 2k\,x^{k-1} + x^k\partial_x\,.$$

Eq. (6) is thus equivalent to

$$(7) \qquad \sum_{j=1}^{n} D_k^{(j)} \, w_n(x_1, \dots, x_n) = 0,$$

where $D_k^{(j)}$ denotes the operator D_k with respect to the variable x_j. Next we insert the recursion relation for w_n derived in §7 and it is then easy to check that (7) reduces to the corresponding equations for w_{n-1} and w_{n-2}. By induction, this proves the claim.

9. The compact picture

The goal here is to show that correlation functions of θ can be analytically extended to $\overline{\mathbb{R}} \cong S^1$, the compactification of $\mathbb{R}$. This a consequence of weak conformal invariance as expressed through eq. (7).

Eq. (7) is the infinitesimal expression of the following symmetry of $w_n(x_1, \dots, x_n)$. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl(2, \mathbb{R})$$

be close to the identity and define

$$g \cdot x = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Then, provided $\gamma x_j + \delta > 0$ for all j, we have

$$w_n(g \cdot x_1, \dots, g \cdot x_n) = \prod_{j=1}^{n} (\gamma x_j + \delta)^4 \, w_n(x_1, \dots, x_n).$$

The obvious problem with this symmetry is that $Sl(2, \mathbb{R})$

does not act in a regular way on $\mathbb{R}$: when $\gamma x + \delta = 0$, $g \cdot x$ is not well-defined. This difficulty has been studied in detail in the 70'ties and a complete and general solution has been found [3]. In the present context, the basic observation is that $Sl(2, \mathbb{R})$ acts in a regular way on

$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

which can be regarded as the one-point compactification of $\mathbb{R}$ through the stereographic projection

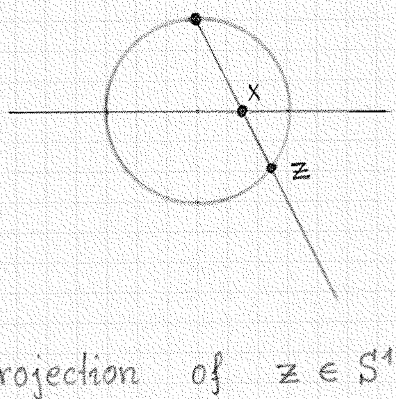
$$z \to x = \operatorname{Re} z \,/\, (1 - \operatorname{Im} z)$$

(cp. figure). If we parametrize z by

$$z = -i e^{i\tau}, \quad -\pi < \tau < \pi,$$

we have

$$x = \operatorname{tg} \frac{\tau}{2}.$$



Stereographic projection of $z \in S^1 \to x \in \mathbb{R}$.

For the action of $g \in Sl(2,\mathbb{R})$ on z, we have

$$g \cdot z = \frac{az+b}{b^* z + a^*},$$

$$a = \tfrac{1}{2}(\alpha + \delta + i\beta - i\gamma),$$

$$b = \tfrac{1}{2}(\beta + \gamma + i\alpha - i\delta),$$

which is manifestly non-singular. Note that for $b = 0$, $g \cdot z$ just amounts to a translation of τ, i.e. the rotations of the circle S^1 are conformal transformations.

Now we lift the energy momentum tensor $\theta(x)$ to a field on S^1 through

$$T(z) = (\cos\tfrac{\tau}{2})^{-4}\,\theta(\mathrm{tg}\,\tfrac{\tau}{2}).$$

Then, and this is a rather non-trivial consequence of conformal invariance [3], this field extends to an operator valued distribution on S^1 such that

$$v_n(z_1, \dots, z_n) = \langle 0 | T(z_1) \dots T(z_n) | 0 \rangle$$

is a boundary value of a function analytic for

$$1 > |z_1| > |z_2| \dots > |z_n| > 0.$$

Furthermore, v_n is globally invariant under $Sl(2,\mathbb{R})$:

$$v_n(g\cdot z_1, \dots, g \cdot z_n) = \prod_{j=1}^{n} |b^* z_j + a^*|^4 \, v_n(z_1, \dots, z_n).$$

Note that this transformation is completely regular; in particular, v_n is invariant under a translation of the variables τ_j.

10. The Virasoro algebra

Eq. (4) now implies

$$[T(z), T(w)] = \frac{8c}{3\pi}\, i^3 \left(\delta'''(\tau - \omega) + \delta'(\tau - \omega) \right)$$

$$+ 16\, i\, \delta'(\tau - \omega)\, T(w) - 8i\, \delta(\tau - \omega)\, \partial_\omega T(w).$$

Thus, if we define the Fourier components L_n through

$$L_n = \frac{1}{8i} \oint_{S^1} dz\, z^{-n-1}\, T(z),$$

one finds that

$$[L_n, L_m] = \frac{c}{12} (n - n^3)\, \delta_{n+m} - (n - m)\, L_{n+m}.$$

This is the Virasoro algebra. Note that the operators L_n can be arbitrarily multiplied since they are obtained by smearing a Wightman field with an admissable testfunction. We obviously have

$$L_n{}^{+} = L_{-n}.$$

Furthermore, from

$$\langle 0 |\, T(z)\, T(w)\, | 0 \rangle = \frac{8c}{\pi^2}\, \frac{z^2 w^2}{(z - w + \varepsilon z)^4}$$

it follows that

$$\langle 0 |\, L_n^{+} L_n\, | 0 \rangle = 0 \quad \text{if} \quad n \leq 1,$$

and hence

(8) $$L_n |0\rangle = 0 \quad \text{for} \quad n \leq 1.$$

For $n \leq -1$ this result can be regarded as a consequence of the spectrum condition: $L_n|0\rangle$ would be a state with negative energy. For $n = 0, \pm 1$, the L_n's are the generators of $Sl(2,\mathbb{R})$ and (8) thus reflects the invariance of the correlation functions $V_n(z_1, \dots, z_n)$.

As is well-known, the operators L_n act as raising and lowering operators with respect to the "conformal Hamiltonian" L_0:

$$[L_0, L_n] = n L_n.$$

The vacuum is thus a lowest weight vector, and the representation of the Virasoro algebra in the vacuum sector is a lowest weight representation with lowest weight $h = 0$.

Suppose now that besides the energy momentum tensor there are other local fields in the theory. Under a few weak additional assumptions one may show that among those fields there are basic fields, called primary, which have the following simple transformation law under the action of θ and $\bar{\theta} = \theta_{--}$:

$$[\theta(x), \phi(y)] = 2ih\, \delta'(x_+ - y_+)\, \phi(y) - 2i\, \delta(x_+ - y_+)\, \partial_+ \phi(y),$$

$$[\bar{\theta}(x), \phi(y)] = 2i\bar{h}\, \delta'(x_- - y_-)\, \phi(y) - 2i\, \delta(x_- - y_-)\, \partial_- \phi(y).$$

Here, h and $\bar{h}$ denote the conformal quantum numbers of ϕ.

The 2-point function of ϕ may then be shown to be given by

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = A \, (x_+ - y_+ - i\varepsilon)^{-2h} \, (x_- - y_- - i\varepsilon)^{-2\bar{h}}$$

where A is some constant. Locality with conventional statistics requires the "spin"

$$s = h - \bar{h}$$

to be integral or half-integral corresponding to Bose respectively Fermi statistics.

Since the correlation functions

$$\langle 0 | \phi(x) \, \theta(x_1) \ldots \theta(x_n) \, \bar{\theta}(y_1) \ldots \bar{\theta}(y_m) \, \phi(y) | 0 \rangle$$

are locally $Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R})$ invariant, we can pass to the compact picture by defining

$$\tau_+ = \tau + \sigma \, , \qquad \tau_- = \tau - \sigma ,$$

$$x_+ = tg \, \frac{\tau_+}{2} \, , \qquad x_- = tg \, \frac{\tau_-}{2} ,$$

$$\Psi(\tau, \sigma) = \left[\cos \frac{\tau_+}{2} \right]^{-2h} \left[\cos \frac{\tau_-}{2} \right]^{-2\bar{h}} \phi(x(\tau, \sigma)) .$$

Ψ extends to a periodic (anti-periodic) field on $\mathbb{R} \times S^1$ depending on whether s is integral or half-integral. Furthermore, the vector

$$|\tau_+, \tau_- \rangle = \Psi(\tau, \sigma) | 0 \rangle$$

extends in both variables to a holomorphic function in the half-planes $\mathrm{Im}\,\tau_+ > 0$, $\mathrm{Im}\,\tau_- > 0$ and it satisfies

$$|\tau_+ + 2\pi, \tau_-\rangle = e^{i 2\pi h} |\tau_+, \tau_-\rangle,$$

$$|\tau_+, \tau_- + 2\pi\rangle = e^{i 2\pi \bar{h}} |\tau_+, \tau_-\rangle.$$

Thus,

$$|\tau_+, \tau_-\rangle = e^{i(h\tau_+ + \bar{h}\tau_-)} \sum_{n_\pm \geq 0} e^{i(n_+\tau_+ + n_-\tau_-)} |n_+, n_-\rangle,$$

where the states $|n_+, n_-\rangle$ are normalizable.

The action of L_n and $\bar{L}_n$ on $|n_+, n_-\rangle$ follows from the commutation rules and is given by

$$L_n |n_+, n_-\rangle = [n_+ + n + (1-n)h] |n_+ + n, n_-\rangle,$$

$$\bar{L}_n |n_+, n_-\rangle = [n_- + n + (1-n)\bar{h}] |n_+, n_- - n\rangle$$

for $n \leq 1$. Thus, $|0,0\rangle$ is a lowest weight vector,

$$L_0 |0,0\rangle = h |0,0\rangle, \qquad \bar{L}_0 |0,0\rangle = \bar{h} |0,0\rangle,$$

$$L_n |0,0\rangle = L_n |0,0\rangle = 0 \quad \text{for} \quad n < 0,$$

and all other states $|n_+, n_-\rangle$ are obtained from $|0,0\rangle$ by applying powers of L_1 and $\bar{L}_1$.

To sum up, we have shown that primary fields give rise to lowest weight representations of the Virasoro algebras L_n, $\bar{L}_n$, the lowest weight being related to the dimension and "spin" of the field.

As far as I know, the converse, namely that for any lowest weight vector there exists an associated primary field, has not been shown. Many people, however, assume that this is the case in the models studied.

Appendix: A model with spontaneous conformal symmetry breaking.

Suppose $\phi(x)$ $(x = x_+ \in \mathbb{R})$ is a free field with two-point function

$$\langle 0 | \phi(x)\, \phi(y) | 0 \rangle = -\frac{1}{2\pi} (x - y - i\varepsilon)^{-2}.$$

The basic commutator is

$$[\phi(x), \phi(y)] = i\, \delta'(x-y).$$

Define

$$\theta(x) = \,:\phi(x)^2: + \lambda\, \partial_x \phi(x),$$

where $\lambda \in \mathbb{R}$ is an arbitrary parameter. Then, it is trivial to verify that the theory with the basic fields $\phi(x)$, $\theta(x)$ satisfies all the requirements listed in § 3 (with $\theta_{++} = \theta$ and $\theta_{--} = \theta_{+-} = 0$). In particular, since

$$[\theta(x), \phi(y)] = i\lambda\, \delta''(x-y) + 2i\, \delta'(x-y)\, \phi(y) - 2i\, \delta(x-y)\, \partial_y \phi(y),$$

the translations are indeed generated by θ. The central charge can be calculated is found to be given by

$$c = 1 + 6\pi \lambda^2.$$

This proves, incidentally, that the representation of the Virasoro algebra in the vacuum sector does not have negative norm states for all $c \geq 1$.

The breaking of the conformal symmetry is now seen by noting

$$\langle 0 | \delta_2 \phi(x) | 0 \rangle = i\lambda ,$$

which does not vanish for $\lambda \neq 0$. Thus, on grounds of the assumptions made in § 3 one cannot establish infinitesimal conformal invariance.

It is conceivable that the spontaneous breakdown of the conformal symmetry is always associated with the existence of a field $\phi(x)$ as above. This could be the "conformal Goldstone theorem".

References

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