Lattice fermions in 4+1 dimensions (addendum)

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This is an extension of ref. [1], covering the case of non-zero physical fermion masses. Moreover some of the basic results are cast into a new form which remains valid in a larger range of m_0 than previously allowed. The notation is taken over completely and we initially assume that m_0 satisfies the bounds (4.3). Only t-independent gauge fields are considered.

11. Massive fermions

As for any other lattice Dirac operator satisfying the Ginsparg-Wilson relation, the natural definition of the massive Dirac operator is

$$D_m = (1 - \tfrac{1}{2}am)D + m, \tag{11.1}$$

where m is the bare mass parameter. In principle m can take any value, also negative ones, but in the following discussion we shall assume that $0 \leq am \leq 2$ for reasons to become clear later.

The massive propagator can be obtained from the functional integral in 4+1 dimensions by adding the term

$$a^4 \sum_x \tfrac{1}{2}am\,\bar{q}(x)q(x) \tag{11.2}$$

to the fermion action (6.3), the boundary fields $q(x)$ and $\bar{q}(x)$ being defined through eqs. (6.1),(6.2). Explicitly this term reads

$$a^4 \sum_x \tfrac{1}{2}am\left\{\overline{\psi}(a_t, x)P_+\psi(T - a_t, x) + \overline{\psi}(T - a_t, x)P_-\psi(a_t, x)\right\}, \tag{11.3}$$

which shows that it is like a hopping term connecting the last time slice of the lattice with the first. The total action is thus given by

$$S_{\rm F} = a_t a^4 \sum_{0<t<T} \sum_{x} \overline{\psi}(t,x)\mathfrak{D}_m\psi(t,x), \tag{11.4}$$

$$\mathfrak{D}_m\psi(t) = \mathfrak{D}\psi(t) + \frac{am}{2a_t}\left\{\delta_{t,a_t}P_+\psi(T-a_t) + \delta_{t,T-a_t}P_-\psi(a_t)\right\}. \tag{11.5}$$

Note that $\mathfrak{D}_m$ acts on the same space of functions as $\mathfrak{D}$, with the same boundary conditions.

It is now evident that

$$\frac{\partial}{\partial m}\langle\psi(t,x)\overline{\psi}(s,y)\rangle = -\tfrac{1}{2}a^5\sum_{z}\langle\psi(t,x)\bar{q}(z)\rangle\langle q(z)\overline{\psi}(s,y)\rangle. \tag{11.6}$$

The inverse of the two-point function $\langle q(x)\bar{q}(y)\rangle$, when interpreted as integral operator in four dimensions, is hence linear in m. Taking eq. (6.11) into account, this implies [2]

$$\langle q(x)\bar{q}(y)\rangle = \frac{2-aD_N}{aD_{m,N}}, \tag{11.7}$$

$$D_{m,N} = (1-\tfrac{1}{2}am)D_N + m. \tag{11.8}$$

An interesting special case is

$$\langle q(x)\bar{q}(y)\rangle|_{am=2} = 1 - \tfrac{1}{2}aD_N \tag{11.9}$$

which shows that the action of D_N on any given source field can be computed by setting $am = 2$. In general we have

$$(1-\tfrac{1}{2}am)\langle q(x)\bar{q}(y)\rangle = -1 + \frac{2}{aD_{m,N}} \tag{11.10}$$

and one thus obtains the massive propagator up to a normalization constant.

The determinant of $\mathfrak{D}_m$ may be worked out similarly. First note that

$$\frac{\partial}{\partial m}\ln\det\mathfrak{D}_m = -\tfrac{1}{2}a^5\sum_{x}\langle\bar{q}(x)q(x)\rangle. \tag{11.11}$$

From eq. (11.7) one infers

$$-\tfrac{1}{2}a^5 \sum_x \langle \bar{q}(x)q(x)\rangle = \operatorname{Tr}\left\{(1-\tfrac{1}{2}aD_N)/D_{m,N}\right\} = \frac{\partial}{\partial m}\ln\det D_{m,N}. \tag{11.12}$$

When combined with eq. (10.1) this yields

$$\det \mathfrak{D}_m = (1/a_t)^{d_{\rm F}} \det\{\tfrac{1}{2}aD_{m,N}\}\det\{1+(RR^\dagger)^N\}(\det B_+)^N. \tag{11.13}$$

All the mass dependence of $\det \mathfrak{D}_m$ thus arises from the factor $\det D_{m,N}$.

12. Alternative expression for D_N

We now rewrite D_N in a different form which allows one to extend the range of m_0 without running into singularities. To this end we introduce the operators

$$K_\pm = \tfrac{1}{2} \pm \tfrac{1}{2}\gamma_5 a_t M(2+a_tM)^{-1}. \tag{12.1}$$

The inverse of $2+a_tM$ is well-defined for $a_tm_0 < 2$, because the spectrum of this operator is then strictly on the right of the imaginary axis. From the definition (12.1) it is immediate that

$$K_+ + K_- = 1, \qquad (K_\pm)^\dagger = K_\pm. \tag{12.2}$$

In particular, K_+ and K_- can be diagonalized simultaneously and have only real eigenvalues.

Next we note that

$$K_+ = \begin{pmatrix} B_+ & C \\ 0 & 1 \end{pmatrix}(2+a_tM)^{-1}, \qquad K_- = \begin{pmatrix} 1 & 0 \\ -C^\dagger & B_- \end{pmatrix}(2+a_tM)^{-1}. \tag{12.3}$$

Recalling eq. (7.1) it is then straighforward to show that

$$RR^\dagger = K_-/K_+. \tag{12.4}$$

Note that K_+ is guaranteed to be invertible if B is, since

$$\det K_+ = \det B_+/\det(2+a_tM). \tag{12.5}$$

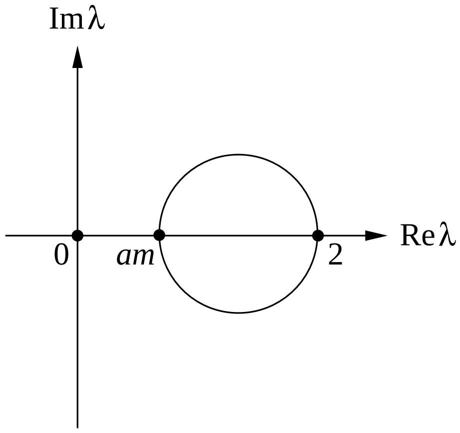

Fig. 2. The eigenvalues λ of $aD_{m,N}$ are contained in a region bounded by a circle in the right half-plane. The radius of the circle decreases linearly from 1 to 0 in the range $0 \leq am \leq 2$.

In particular, the representation (12.4) is valid in the parameter range (4.3).

The operator D_N may now be rewritten in the form

$$aD_N = 1 + \gamma_5 \frac{K_+^N - K_-^N}{K_+^N + K_-^N}. \tag{12.6}$$

The important point here is that this expression is manifestly analytic in m_0 in the extended range

$$m_0 > 0, \qquad a_t m_0 < 2, \qquad a m_0 < 2. \tag{12.7}$$

The right-hand side of

$$\det \mathfrak{D}_m = (1/a_t)^{d_{\rm F}} \det\{\tfrac{1}{2}aD_{m,N}\} \det\{K_+^N + K_-^N\} \det(2 + a_t M)^N \tag{12.8}$$

thus has to be equal to $\det \mathfrak{D}_m$ everywhere in this range.

From eq. (12.6) one also infers that $\|aD_N - 1\| \leq 1$. The spectrum of $aD_{m,N}$ is hence confined to the circular region shown in fig. 2. In particular, zero-modes are excluded for $m > 0$. Taking eq. (12.8) into account, one concludes from this that $\mathfrak{D}_m$ is invertible in the extended range of parameters. Eqs. (11.7)–(11.10) thus remain valid in this range, provided the new expression is substituted for D_N.

For $m \leq 0$ it can happen that $D_{m,N}$ has a zero-mode. Since $\mathfrak{D}_m$ is singular under these conditions, it is clear that this leads to various technical complications that should better be avoided. This is a limitation of the domain-wall fermion approach. D_m itself is well-defined and invertible also for $m < 0$.

13. Large N limit revisited

The limit $N \to \infty$ can easily be taken at all points in the range (12.7) by noting that the simultaneous eigenvalues of $K_\pm$ are of the form $\frac{1}{2}(1 \pm \nu)$ with $\nu \in \mathbb{R}$. As a result one obtains

$$aD \equiv \lim_{N\to\infty} aD_N = 1 + \gamma_5\,\epsilon(K_+ - K_-), \tag{13.1}$$

which is equivalent to

$$aD = 1 - A(A^\dagger A)^{-1/2}, \tag{13.2}$$

$$A = -a_t M(2 + a_t M)^{-1}. \tag{13.3}$$

Compared to Neuberger's operator, the only difference thus consists in the factor $(2 + a_t M)^{-1}$ in the definition of A. Since this factor is local, bounded and has its spectrum strictly on the right of the imaginary axis, the locality of D can again be proved for all gauge fields with plaquette loops close to 1.

The large N limit is approached with an exponential rate

$$\omega = \ln \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}, \tag{13.4}$$

where α is the smallest eigenvalue of $A^\dagger A$. The corresponding generalized eigenvalue equation reads

$$a_t^2 M^\dagger M\psi = \alpha\,(2 + a_t M)^\dagger (2 + a_t M)\psi. \tag{13.5}$$

This is a well-posed problem in the parameter range (12.7), since the operator on the right-hand side is guaranteed to be strictly positive.

14. Accelerating the convergence at large N

So far we have assumed that M is of the form (4.2), but the final results quoted above [eqs. (12.6),(12.8),(13.1)–(13.5) with $K_{\pm}$ given by eq. (12.1)] are actually valid for any operator M satisfying

$$M^{\dagger} = \gamma_5 M \gamma_5, \qquad \det(2 + a_t M) \neq 0. \tag{14.1}$$

This can be shown by going back to the general formulae for the determinant and the Green function of $\mathfrak{D}$ given in sects. 2,3 and by working them out directly in terms of $K_{\pm}$. The solution matrix $S(t)$, for example, is given by

$$S(t) = (1 + P_{+} a_t M)^{-1} (K_{-}/K_{+})^{t/a_t - 1} P_{-} \tag{14.2}$$

and similar expressions are obtained for the other fundamental solutions (as before one first considers the case where B and thus $K_{\pm}$ are non-singular).

One can make use of this fact to accelerate the convergence at large N by replacing $a_t M$ through an operator of the form

$$a_t \hat{M} = a_t M - \sum_{k,l=1}^{r} X_{kl}\, w_k \otimes w_l^{\dagger} \gamma_5. \tag{14.3}$$

The idea is to choose the vectors w_k and the hermitian matrix X_{kl} so that the smallest eigenvalues λ_k of $\gamma_5 A$ are replaced by larger values $\hat{\lambda}_k$ while all other eigenvalues are unchanged. In this way the exponent ω characterizing the approach to the large N limit can be significantly increased with a modest computational effort. Evidently all this is very similar to the acceleration method of ref. [3,4] previously employed in the case of Neuberger's operator.

So let us suppose that

$$\gamma_5 A v_k = \lambda_k v_k, \qquad k = 1, \ldots, r, \qquad (v_k, v_l) = \delta_{kl}, \tag{14.4}$$

where $r \geq 1$ is any fixed integer. If we set

$$w_k = (2 + a_t M) \gamma_5 v_k, \tag{14.5}$$

$$(X^{-1})_{kl} = 2\delta_{kl} (\hat{\lambda}_k - \lambda_k)^{-1} + \big(v_k, (2 + a_t M) \gamma_5 v_l\big), \tag{14.6}$$

a short calculation yields

$$\hat{A} \equiv -a_t\hat{M}(2+a_t\hat{M})^{-1} = A + \sum_{k=1}^{r} (\hat{\lambda}_k - \lambda_k)\gamma_5 v_k \otimes v_k^{\dagger}. \tag{14.7}$$

The operator $\gamma_5\hat{A}$ has thus the same eigenvectors as $\gamma_5 A$, with the same eigenvalues except for those associated with the eigenvectors v_k which are equal to $\hat{\lambda}_k$ instead of λ_k.

It follows from this that the corresponding operators $\hat{D}_N$ and D_N converge to the same Dirac operator D if

$$\epsilon(\hat{\lambda}_k) = \epsilon(\lambda_k) \quad \text{for all} \quad k = 1,\ldots,r. \tag{14.8}$$

A possible choice of $\hat{\lambda}_k$ is thus

$$\hat{\lambda}_k = \epsilon(\lambda_k), \tag{14.9}$$

which implies instantaneous convergence of $\hat{D}_N$ on the subspace spanned by the eigenvectors v_k. One should however make sure that the matrix on the right-hand side of eq. (14.6) is well-conditioned. There is enough flexibility in the choice of $\hat{\lambda}_k$ to achieve this without giving up the improved convergence properties of $\hat{D}_N$.

References

[1] M. Lüscher, Lattice fermions in 4+1 dimensions, notes (August 1999)

[2] G. de Divitiis, notes (November 1999)

[3] P. Hernández, K. Jansen, L. Lellouch, Phys. Lett. B469 (1999) 198

[4] P. Hernández, K. Jansen, L. Lellouch, A numerical treatment of Neuberger's lattice Dirac operator, hep-lat/0001008