Lattice fermions in 4+1 dimensions

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1. Introduction

In this note we consider Dirac fermions on a hyper-cubic lattice in 4+1 dimensions with Schrödinger functional boundary conditions. The aim is to derive some general formulae for the fermion determinant and the propagator, which allow one to study the reduction to an effective chiral theory in 4 dimensions in a mathematically controlled way. Eventually one would like to deduce the expression for the chiral determinant obtained in ref. [1] from the fermion determinant in 4+1 dimensions.

The lattice Dirac operator is assumed to be of the form

$$\mathfrak{D} = \tfrac{1}{2}\left\{\gamma_5(\partial_t^* + \partial_t) - a_t\partial_t^*\partial_t\right\} + M(t), \tag{1.1}$$

where t and a_t denote the coordinate and the lattice spacing in the fifth direction. The possible values of t are

$$t = 0, a_t, 2a_t, \ldots, T \tag{1.2}$$

and the forward and backward difference operators ∂_t and ∂_t^* are defined as usual. We do not make any assumptions on the operator $M(t)$ at this point except that

$$B(t) = 1 + \tfrac{1}{2}a_t\left[M(t) + \gamma_5 M(t)\gamma_5\right] \tag{1.3}$$

should be invertible. Later it will be set to the 4-dimensional massive Dirac operator with a t-dependent gauge field.

The dynamical degrees of freedom of the fermion field $\psi(t,x)$ reside on the lattice sites with $0 < t < T$. At the boundaries only half of the Dirac components are defined and these are fixed through

$$P_+\psi(0,x) = P_-\psi(T,x) = 0, \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5). \tag{1.4}$$

For brevity we shall often omit to indicate the position on the 4-dimensional lattice, which is, incidentally, always assumed to be finite with suitable boundary conditions. With these specifications the action of the Dirac operator,

$$\mathfrak{D}\psi(t) = (1/a_t)\left\{-P_-\psi(t+a_t) - P_+\psi(t-a_t) + [1 + a_t M(t)]\,\psi(t)\right\}, \tag{1.5}$$

is well-defined for all t in the range $0 < t < T$ and $\mathfrak{D}$ may hence be considered to be linear mapping from the space of fermion fields into itself. Note that the dimension $d_{\rm F}$ of this space is finite. With respect to any particular basis the Dirac operator is, therefore, just a complex $d_{\rm F} \times d_{\rm F}$ matrix.

2. Determinant of $\mathfrak{D}$

The determinant of ordinary differential operators may be expressed through a particular solution of the associated homogeneous differential equation. This is a well-known fact, which has previously been exploited when discussing the quantum fluctuations around classical field configurations, for example [2]. Essentially the same remark applies to ordinary difference operators and the determinant of $\mathfrak{D}$ can thus be computed along these lines.

The general solution of the homogeneous problem

$$\mathfrak{D}\psi(t) = 0, \qquad t > 0, \qquad P_+\psi(0) = 0, \tag{2.1}$$

may be obtained straightforwardly by rewriting eq. (1.5) in the form

$$P_+\psi(t) = B(t)^{-1}P_+\left\{\psi(t-a_t) - a_t M(t)P_-\psi(t)\right\}, \tag{2.2}$$

$$P_-\psi(t+a_t) = B(t)P_-\psi(t) + P_- a_t M(t)P_+\psi(t). \tag{2.3}$$

For given initial value $P_-\psi(a_t)$, the first of these equations determines $P_+\psi(a_t)$ and the second may then be applied to compute $P_-\psi(2a_t)$. Proceeding in this way one generates the complete solution which is, therefore, uniquely determined. Since the solution depends linearly on the initial data, we have

$$\psi(t) = S(t)P_-\psi(a_t), \qquad S(t)P_- = S(t), \tag{2.4}$$

where $S(t)$ is an operator acting on fermion fields in four dimensions.

The determinant of the Dirac operator is now given by

$$\det \mathfrak{D} = (1/a_t)^{d_{\rm F}} \det\bigl\{P_+ + P_-S(T)\bigr\} \prod_{s=a_t}^{T-a_t} \det\bigl\{P_- + B(s)P_+\bigr\}. \tag{2.5}$$

It is not difficult to prove this formula, but the argument is somewhat lengthy and is therefore deferred to appendix A.

3. Propagator

The Green function $\mathfrak{G}(t,s)$ associated with the Dirac operator $\mathfrak{D}$ is defined through

$$\mathfrak{D}\mathfrak{G}(t,s) = a_t^{-1}\delta_{ts}, \qquad 0 < t,s < T, \tag{3.1}$$

where fermion boundary conditions

$$P_+\mathfrak{G}(0,s) = P_-\mathfrak{G}(T,s) = 0 \tag{3.2}$$

are assumed. $\mathfrak{G}(t,s)$ is the kernel in position space of the inverse $\mathfrak{D}^{-1}$ of the Dirac operator. In particular, it is uniquely determined through eqs. (3.1) and (3.2) if $\mathfrak{D}$ has no zero-modes.

As in the case of the determinant of $\mathfrak{D}$, the Green function can be expressed through the solutions of the homogeneous Dirac equation. Since the Dirac operator is not hermitean, one also needs to consider the operator

$$\widetilde{\mathfrak{D}} = \tfrac{1}{2}\left\{-\gamma_5(\partial_t^* + \partial_t) - a_t\partial_t^*\partial_t\right\} + M(t)^\dagger \tag{3.3}$$

to be able to describe the s-dependence of $\mathfrak{G}(t,s)$. One then has two independent solutions per operator (table 1). Similarly to $S(t)$ they can all be constructed through a two-step recursion, starting from the initial values specified in the table.

Table 1. Fundamental solutions of the Dirac equation

$\mathfrak{D}S(t) = 0 \quad (t > 0),$	$P_+S(0) = 0,$	$P_-S(a_t) = P_-$
$\widetilde{\mathfrak{D}}\widetilde{S}(t) = 0 \quad (t > 0),$	$P_-\widetilde{S}(0) = 0,$	$P_+\widetilde{S}(a_t) = P_+$
$\mathfrak{D}S'(t) = 0 \quad (t < T),$	$P_-S'(T) = 0,$	$P_+S'(T - a_t) = P_+$
$\widetilde{\mathfrak{D}}\widetilde{S}'(t) = 0 \quad (t < T),$	$P_+\widetilde{S}'(T) = 0,$	$P_-\widetilde{S}'(T - a_t) = P_-$

An important fact which we shall make use of is that the Wronskian

$$W(\chi, \psi) = \chi(t + a_t)^\dagger P_+\psi(t) - \chi(t)^\dagger P_-\psi(t + a_t) \tag{3.4}$$

of any two fermion fields satisfying

$$\widetilde{\mathfrak{D}}\chi(t) = 0, \qquad \mathfrak{D}\psi(t) = 0, \tag{3.5}$$

is independent of t. This can be shown straightforwardly by inserting the Dirac equations in the definition (3.4). Incidentally, when only the P_+ or P_- projected equations are inserted, one finds that the Wronskian may also be written in the form

$$W(\chi, \psi) = \chi(t)^\dagger \gamma_5 B(t)\psi(t). \tag{3.6}$$

The Wronskians of the fundamental solutions listed in table 1 are given by

$$W(\widetilde{S}, S) = W(\widetilde{S}', S') = 0, \tag{3.7}$$

$$W(\widetilde{S}, S') = P_+S'(0), \tag{3.8}$$

$$W(\widetilde{S}', S) = -P_-S(T), \tag{3.9}$$

as may easily be shown by evaluating eq. (3.4) at the boundaries of the lattice.

To construct the Green function, we start from the ansatz

$$\mathfrak{G}(t, s) \underset{t>s}{=} S'(t)X'(s), \qquad P_+X'(s) = X'(s), \tag{3.10}$$

$$\mathfrak{G}(t,s) \underset{t<s}{=} S(t)X(t), \qquad P_-X(s) = X(s), \tag{3.11}$$

$$\mathfrak{G}(s,s) = P_+S'(s)X'(s) + P_-S(s)X(s), \tag{3.12}$$

where $X(s)$ and $X'(s)$ are to be determined. This ansatz satisfies the boundary conditions (3.2) and also the Dirac equation (3.1) for all $t \neq s$. The equation should also hold at $t = s$ which will be the case if and only if

$$\gamma_5 B(s)\left\{S'(s)X'(s) - S(s)X(s)\right\} = 1. \tag{3.13}$$

Using the Wronskian, eqs. (3.6)–(3.9), this condition can easily be solved for $X(s)$ and $X'(s)$ and one then ends up with the result

$$\mathfrak{G}(t,s) \underset{t>s}{=} S'(t)\left\{P_- + P_+S'(0)\right\}^{-1}\widetilde{S}(s)^\dagger, \tag{3.14}$$

$$\mathfrak{G}(t,s) \underset{t<s}{=} S(t)\left\{P_+ + P_-S(T)\right\}^{-1}\widetilde{S}'(s)^\dagger, \tag{3.15}$$

$$\begin{aligned}\mathfrak{G}(s,s) = {}& P_+S'(s)\left\{P_- + P_+S'(0)\right\}^{-1}\widetilde{S}(s)^\dagger \\ & + P_-S(s)\left\{P_+ + P_-S(T)\right\}^{-1}\widetilde{S}'(s)^\dagger.\end{aligned} \tag{3.16}$$

Note that the inverse of the operators in the curly brackets exists if $\mathfrak{D}$ has no zero modes, i.e. if the Green function is well-defined.

4. Standard choice of $M(t)$

On a 4-dimensional lattice with lattice spacing a, the Wilson-Dirac operator is defined as usual through

$$D_{\rm w} = \tfrac{1}{2}\left\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^*\nabla_\mu\right\}, \tag{4.1}$$

with ∇_μ and ∇_μ^* being the gauge-covariant forward and backward difference operators. The conventional choice for $M(t)$ is

$$M(t) = D_{\rm w} - m_0, \tag{4.2}$$

where it is understood that a t-dependent gauge field is inserted in the covariant derivatives. The mass parameter m_0 has to be such that

$$m_0 > 0, \qquad a_t m_0 < 1, \qquad a m_0 < 2, \tag{4.3}$$

in order to obtain the desired behaviour of the theory in 4+1 dimensions. Without further notice only this choice of $M(t)$ will be considered in the following and we shall always take it for granted that the inequalities (4.3) are satisfied.

In a chiral representation of the Dirac matrices (appendix B) we have

$$B(t) = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \qquad 1 + a_t M(t) = \begin{pmatrix} B_+ & C \\ -C^\dagger & B_- \end{pmatrix}, \tag{4.4}$$

the block operators being given by

$$B_\pm = 1 - a_t m_0 - \tfrac{1}{2} a_t a \nabla_\mu^* \nabla_\mu, \tag{4.5}$$

$$C = \tfrac{1}{2} a_t e_\mu (\nabla_\mu^* + \nabla_\mu) \tag{4.6}$$

Note that $B(t)$ is bounded from below by $1 - a_t m_0$ and is thus strictly positive. In particular, $B(t)$ is invertible for any choice of the gauge field.

5. Determinant and propagator for constant fields

If $M(t)$ does not depend on t, the fundamental solutions of the Dirac equation defined in table 1 can be worked out analytically. The key observation is that the two equations

$$P_- \mathfrak{D} \psi(t) = 0, \qquad P_+ \mathfrak{D} \psi(t + a_t) = 0, \tag{5.1}$$

are equivalent to

$$B^{1/2} \psi(t + a_t) = R^\dagger R \, B^{1/2} \psi(t), \tag{5.2}$$

where the operator R is given by

$$R = \begin{pmatrix} 1 & 0 \\ -C^\dagger & 1 \end{pmatrix} \begin{pmatrix} (B_+)^{-1/2} & 0 \\ 0 & (B_-)^{1/2} \end{pmatrix}. \tag{5.3}$$

An important point to note is that the equivalence holds at any specified value of t. This allows one to easily control the situation close to the boundaries of the lattice and with little additional work one finds that

$$S(t) = B^{-1/2}(R^\dagger R)^{t/a_t} B^{-1/2} P_-, \tag{5.4}$$

$$S'(t) = B^{-1/2}(R^\dagger R)^{(t-T)/a_t} B^{-1/2} P_+, \tag{5.5}$$

$$\widetilde{S}(t) = \gamma_5 S'(T-t)\gamma_5, \qquad \widetilde{S}' = \gamma_5 S(T-t)\gamma_5. \tag{5.6}$$

For the determinant of $\mathfrak{D}$ the formula

$$\det \mathfrak{D} = (1/a_t)^{d_{\rm F}} \det\bigl\{P_+ + (R^\dagger R)^{T/a_t} P_-\bigr\}(\det B_+)^{T/a_t - 1}(\det B_-)^{-1} \tag{5.7}$$

is thus obtained (the last two factors could be combined into one since the determinants of B_+ and B_- coincide).

When eqs. (5.4)–(5.6) are inserted in the general expression (3.14)–(3.16) for the Green function $\mathfrak{G}(t,s)$, a few simple manipulations lead to the result

$$\begin{aligned}\mathfrak{G}(t,s) \underset{t>s}{=} & \, B^{-1/2}(R^\dagger R)^{(t-T)/a_t} P_+ \\ & \times \bigl\{(R^\dagger R)^{-T/a_t} P_+ + P_-\bigr\}^{-1}(R^\dagger R)^{-s/a_t} B^{-1/2}\gamma_5,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\mathfrak{G}(t,s) \underset{t<s}{=} & \, -B^{-1/2}(R^\dagger R)^{t/a_t} P_- \\ & \times \bigl\{(R^\dagger R)^{-T/a_t} P_+ + P_-\bigr\}^{-1}(R^\dagger R)^{-s/a_t} B^{-1/2}\gamma_5,\end{aligned} \tag{5.9}$$

$$\begin{aligned}\mathfrak{G}(s,s) = & \, B^{-1/2}\bigl\{P_+(R^\dagger R)^{(s-T)/a_t} P_+ - P_-(R^\dagger R)^{s/a_t} P_-\bigr\} \\ & \times \bigl\{(R^\dagger R)^{-T/a_t} P_+ + P_-\bigr\}^{-1}(R^\dagger R)^{-s/a_t} B^{-1/2}\gamma_5.\end{aligned} \tag{5.10}$$

These expressions completely agree with the formulae obtained by Giulia [4]. Note that the inverse of the operator in the curly bracket on the second lines in these formulae is well-defined, because $R^\dagger R$ is invertible and strictly positive. This shows incidentally that $\mathfrak{D}$ has no zero-modes if the gauge field is t-independent.

6. Boundary field propagator

We now proceed to derive a result of Kikukawa and Noguchi [5] for the two-point function of the boundary fermion fields. The gauge field is again chosen to be independent of t and the boundary fields are defined through

$$q(x) = P_-\psi(a_t, x) + P_+\psi(T - a_t, x), \tag{6.1}$$

$$\bar{q}(x) = \overline{\psi}(a_t, x)P_+ + \overline{\psi}(T - a_t, x)P_-. \tag{6.2}$$

Note incidentally that up to contact terms this definition coincides with the combinations $\zeta(x) + \zeta'(x)$ and $\bar{\zeta}(x) + \bar{\zeta}'(x)$ of the boundary fields considered in the framework of the Schrödinger functional (see ref. [6], sect. 4, for example).

If we set up the functional integral for fermions in 4+1 dimensions in the obvious way, with fermion action

$$S_{\rm F} = a_t a^4 \sum_{0<t<T} \sum_x \overline{\psi}(t, x)\mathfrak{D}\psi(t, x), \tag{6.3}$$

the basic two-point function is given by

$$\langle \psi(t, x)\overline{\psi}(s, y)\rangle = \mathfrak{G}(t, x; s, y). \tag{6.4}$$

For the boundary fields we thus obtain

$$\begin{aligned} \langle q(x)\bar{q}(y)\rangle = {} & P_-\mathfrak{G}(a_t, a_t)P_+ + P_+\mathfrak{G}(T - a_t, T - a_t)P_- \\ & + P_-\mathfrak{G}(a_t, T - a_t)P_- + P_+\mathfrak{G}(T - a_t, a_t)P_+, \end{aligned} \tag{6.5}$$

where, for notational simplicity, the coordinates x and y have been suppressed on the right-hand side. Inserting eqs. (5.8)–(5.10) and using the identities

$$RP_- = P_-B^{1/2}, \qquad R^\dagger P_+ = P_+B^{-1/2}, \tag{6.6}$$

this evaluates to

$$\begin{aligned} \langle q(x)\bar{q}(y)\rangle = {} & B^{1/2}\bigl\{(R^\dagger R)^{-T/a_t}P_+ - P_-\bigr\}^{-1} \\ & \times \bigl\{(R^\dagger R)^{-1}P_+ - (R^\dagger R)^{1-T/a_t}P_-\bigr\}B^{-1/2}. \end{aligned} \tag{6.7}$$

Applying the relation

$$(R^\dagger R)^n = R^\dagger (RR^\dagger)^{n-1} R \tag{6.8}$$

and eq. (6.6) again one then ends up with the expression

$$\langle q(x)\bar{q}(y)\rangle = \bigl\{P_+ - (RR^\dagger)^N P_-\bigr\}^{-1}\bigl\{(RR^\dagger)^N P_+ - P_-\bigr\} \tag{6.9}$$

where $N = T/a_t - 1$. If we introduce an operator D_N through

$$aD_N = 1 - \gamma_5 \frac{(RR^\dagger)^N - 1}{(RR^\dagger)^N + 1}, \tag{6.10}$$

the result of Kikukawa and Noguchi [5],

$$\langle q(x)\bar{q}(y)\rangle = -1 + \frac{2}{aD_N}, \tag{6.11}$$

is thus obtained. Note that it is $RR^\dagger$ and not $R^\dagger R$ which appears in eq. (6.10).

7. Properties of $RR^\dagger$ and the limit $N \to \infty$

The operator

$$RR^\dagger = \begin{pmatrix} 1 & 0 \\ -C^\dagger & 1 \end{pmatrix} \begin{pmatrix} B_+^{-1} & 0 \\ 0 & B_- \end{pmatrix} \begin{pmatrix} 1 & -C \\ 0 & 1 \end{pmatrix} \tag{7.1}$$

has all its eigenvalues in a compact range on the positive real axis. In particular, zero-modes are excluded since

$$\det RR^\dagger = 1. \tag{7.2}$$

Another important identity is

$$RR^\dagger + (RR^\dagger)^{-1} = 2 + a_t^2 M^\dagger B^{-1} M. \tag{7.3}$$

It follows from this that eigenvalues equal to 1 will occur if and only if M has zero-modes. As discussed in ref. [7] this is rarely the case. Moreover a lower bound on $M^\dagger M$ can be established rigorously if the gauge field has plaquette loops $U(p)$ satisfying

$$\|1 - U(p)\| < \epsilon \tag{7.4}$$

for all plaquettes p and some sufficiently small constant ϵ (details are given in ref. [7]). Eq. (7.3) then implies that the spectrum of $RR^\dagger$ has a gap around 1.

It is now evident that D_N has a limit D at large N given by

$$aD = 1 - \gamma_5\{\hat{P}_+ - \hat{P}_-\}, \tag{7.5}$$

where $\hat{P}_\pm$ are the projectors to the subspaces of eigenvectors of $RR^\dagger$ with eigenvalues greater and smaller than 1. This equation may also be written in the form

$$\hat{P}_\pm = \tfrac{1}{2}(1 \pm \hat{\gamma}_5), \qquad \hat{\gamma}_5 = \gamma_5(1 - aD), \tag{7.6}$$

and it is then not difficult to show that D satisfies the Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D. \tag{7.7}$$

In the limit $N \to \infty$ the two-point function of the boundary fields thus yields the inverse of D up to a constant additive term.

As an aside we note that D_N converges to D exponentially fast. The leading error term is proportional to

$$\left\{1 + \tfrac{1}{2}\delta + \sqrt{\delta + \tfrac{1}{4}\delta^2}\right\}^{-N} \tag{7.8}$$

with δ being the lowest eigenvalue of the operator

$$\Delta = a_t^2 M^\dagger B^{-1} M. \tag{7.9}$$

The gap δ can be computed numerically by applying the Ritz functional method of refs. [8,9]. An important technical point to note here is that the spectrum of Δ coincides with the spectrum of the generalized eigenvalue problem

$$a_t^2 M M^\dagger z = \lambda B z. \tag{7.10}$$

The appropriate Ritz functional to consider is thus

$$\mu(z) = (z, a_t^2 M M^\dagger z)/(z, Bz) \tag{7.11}$$

and the minimization then proceeds essentially as described in the papers quoted above. Evidently the advantage of this formulation is that one is concerned with simple difference operators only. In particular, the inverse of B is not required.

8. Locality of D

Since B is bounded from below by $1-a_t m_0$, the techniques of ref. [7] may be applied to show that B^{-1} is a local operator with exponentially decaying tails. The same is hence also true for $RR^\dagger$ and $(RR^\dagger)^{-1}$.

To establish the locality of D we note that

$$aD = 1 - A(A^\dagger A)^{-1/2}, \tag{8.1}$$

a possible choice for A being

$$A = \tfrac{1}{2}\gamma_5\bigl\{RR^\dagger - (RR^\dagger)^{-1}\bigr\}. \tag{8.2}$$

The square root in eq. (8.1) is a potential source of non-locality, but since

$$A^\dagger A = \Delta + \tfrac{1}{4}\Delta^2, \tag{8.3}$$

the operator is in fact local (with exponentially decaying tails) as long as the spectrum of Δ is separated from zero by positive gap δ [7]. Evidently all this is very similar to the situation encountered in the case of Neuberger's operator which coincides with D in the limit $a_t \to 0$.

As already pointed out in ref. [5], the effective action associated with the boundary fields $q(x)$ and $\bar{q}(x)$ is non-local due to the presence of the constant -1 in the propagator (6.11). One can remove this term by redefining the fermion action in 4+1 dimensions according to

$$S_{\rm F} = a_t a^4 \sum_{0<t<T} \sum_x \overline{\psi}(t,x)\mathfrak{D}\psi(t,x) \\ + a^4 \sum_x \bigl\{\overline{\chi}(x)P_+\chi(x) + \overline{\chi}(x)P_-\chi(x)\bigr\}. \tag{8.4}$$

The additional fields $P_+\chi(x)$ and $P_-\chi(x)$ should be thought of as being attached to the boundaries at $t=0$ and $t=T$ respectively. Since they are only affecting the theory at scales of the cutoff, one is certainly free to include them. Now if we define

$$q(x) = (a/2)^{1/2}\{P_-\psi(a_t,x) + P_+\psi(T-a_t,x) + \chi(x)\}, \tag{8.5}$$

$$\bar{q}(x) = (a/2)^{1/2}\{\overline{\psi}(a_t,x)P_+ + \overline{\psi}(T-a_t,x)P_- + \overline{\chi}(x)\}, \tag{8.6}$$

it follows that

$$\langle q(x)\bar{q}(y)\rangle = 1/D_N. \tag{8.7}$$

In the limit $N\to\infty$ the associated effective action

$$S_{\rm eff} = a^4 \sum_x \bar{q}(x) D q(x) \tag{8.8}$$

has then the desired local form. Note that the field $\chi(x)$ plays a rôle similar to the auxiliary fields introduced in ref. [10].

9. Free fermions

In the free case and if acting on plane waves with momentum p, the operator $RR^\dagger$ is given by †

$$RR^\dagger = b^{-1}\{ia_t\mathring{p}_\mu\gamma_\mu\gamma_5 + (1-q)\gamma_5 + q\}, \tag{9.1}$$

$$b = 1 - a_t m_0 + \tfrac{1}{2}a_t a\hat{p}^2, \tag{9.2}$$

$$q = \tfrac{1}{2}\{1 + b^2 + a_t^2\mathring{p}^2\}. \tag{9.3}$$

In particular, its eigenvalues are

$$\lambda_\pm = q/b \pm \sqrt{(q/b)^2 - 1} \tag{9.4}$$

† The standard notations $\hat{p}_\mu = (2/a)\sin(ap_\mu/2)$ and $\mathring{p}_\mu = (1/a)\sin(ap_\mu)$ are employed here

with each of them being doubly degenerate. Note that

$$\lambda_+ > 1, \qquad \lambda_- < 1, \qquad \lambda_+\lambda_- = 1, \tag{9.5}$$

for all momenta p. The total number of eigenvalues greater than 1 is, therefore, precisely half the dimension of the space of fermion fields in 4 dimensions.

For free fermions the Dirac operator defined in sect. 7 can be worked out explicitly and one finds that

$$aD = 1 + \frac{ia_t\mathring{p}_\mu\gamma_\mu + q - 1}{\sqrt{a_t^2\mathring{p}^2 + (q-1)^2}} \tag{9.6}$$

in momentum space. It is easy to check that the argument of the square root in this expression is positive, for all real momenta p. The right-hand side of eq. (9.6) is, therefore, everywhere smooth in the Brillouin zone. At small momenta we have

$$aD = c\left\{ip_\mu\gamma_\mu + \mathrm{O}(p^2)\right\}, \qquad c = \left(m_0 - \tfrac{1}{2}a_t m_0^2\right)^{-1}, \tag{9.7}$$

and for the inverse of the operator one obtains

$$(aD)^{-1} = \frac{1}{2}\left\{1 - \frac{ia_t\mathring{p}_\mu\gamma_\mu}{q - 1 + \sqrt{a_t^2\mathring{p}^2 + (q-1)^2}}\right\}. \tag{9.8}$$

The denominator in this formula vanishes if $p_\mu = 0 \bmod 2\pi/a$ but nowhere else, i.e. there are no doubler modes. In the free case D is hence a perfectly acceptable lattice Dirac operator.

In practice the parameters $a_t m_0$ and am_0 should be chosen so as to achieve good convergence at large N, a small localization radius and small cutoff effects in the energy-momentum relation. We now examine these condition one by one.

a. As discussed in sect. 7 the operator D_N converges to D with a rate given by the gap around 1 of the spectrum of $RR^\dagger$. More precisely, if we define $\omega = \min_p \ln\lambda_+$, the leading error term at large N is proportional to $\mathrm{e}^{-N\omega}$. From eq. (9.4) we have

$$\cosh\omega = \min_p\,(q/b) \tag{9.9}$$

and it is then possible to prove (appendix C) that

$$\mathrm{e}^\omega = \begin{cases} (1 - a_t m_0)^{-1} & \text{if } a_t m_0 \leq 1 + a_t/a - \sqrt{1 + (a_t/a)^2}, \\[1ex] 1 - a_t m_0 + 2a_t/a & \text{otherwise.} \end{cases} \tag{9.10}$$

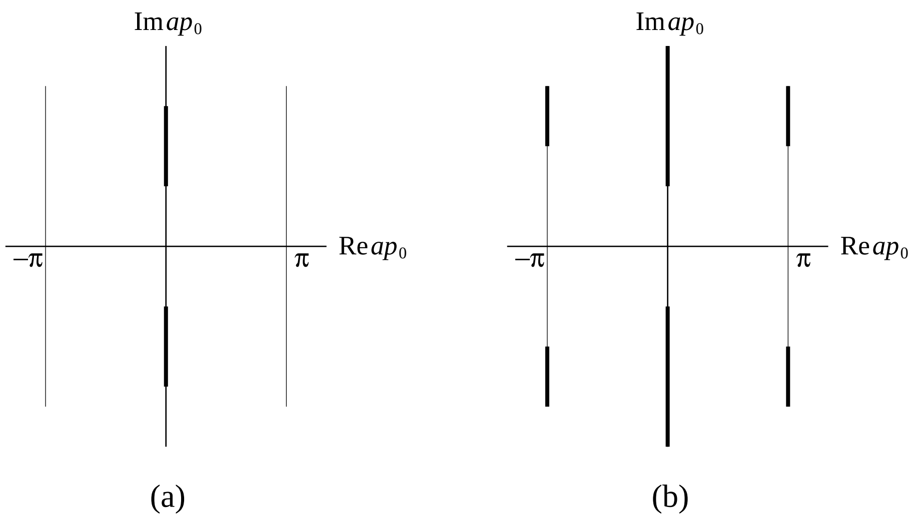

Fig. 1. Analyticity properties of the right-hand side of eq. (9.6) as a function of p_0 at fixed spatial momenta. The thick lines represent cut singularities. Depending on the value of m_0 and the spatial momenta, there is one pair of cuts on the imaginary axis (figure a) or two pairs of cuts at $\mathrm{Re}\, p_0 = 0$ and $\mathrm{Re}\, p_0 = \pi/a$ (figure b).

At fixed a_t/a the fastest convergence is obtained for

$$a_t m_0 = 1 + a_t/a - \sqrt{1 + (a_t/a)^2}. \tag{9.11}$$

This value of m_0 always lies in the allowed range and for the exponent ω one gets

$$\mathrm{e}^{\omega} = a_t/a + \sqrt{1 + (a_t/a)^2}. \tag{9.12}$$

Note that ω can be made arbitrarily large by choosing a_t to be greater than a.

b. Along the time axis the position space kernel $D(x, y)$ of the Dirac operator decays exponentially,

$$D(x, y) \propto \mathrm{e}^{-\epsilon |x_0 - y_0|/a}, \tag{9.13}$$

where ϵ is the minimum distance from the real line of the singularities of the right-hand side of eq. (9.6) in the complex p_0-plane. The singularities arise from the fact that the argument of the square root vanishes at some points. It is possible to work this out explicitly (appendix D) and for the singularity structure the result shown in fig. 1 is then obtained.

The exponent ϵ is equal to the minimum over all spatial momenta of the distance of the cuts from the real axis. This is also worked out in appendix D, the result being

$$\mathrm{e}^{\epsilon} \underset{a_t \leq a}{=} \begin{cases} (1-am_0)^{-1} & \text{if } am_0 \leq 2-\sqrt{2}, \\ 3-am_0 & \text{otherwise,} \end{cases} \tag{9.14}$$

$$\mathrm{e}^{\epsilon} \underset{a_t \geq a}{=} \begin{cases} (1-am_0)^{-1} & \text{if } am_0 \leq 1+a/a_t-\sqrt{1+(a/a_t)^2}, \\ 1-am_0+2a/a_t & \text{otherwise.} \end{cases} \tag{9.15}$$

The absolute maximum, $\epsilon = \ln(1+\sqrt{2})$, is obtained when $a_t \leq a$ and $am_0 = 2-\sqrt{2}$. In particular, it cannot be made arbitrarily large.

c. The one-particle energy E is determined by the pole of the propagator (9.8) in the complex p_0-plane closest to the real axis. At this point we have

$$q-1 = -\sqrt{a_t^2 \mathring{p}^2 + (q-1)^2}, \tag{9.16}$$

where it is understood that the square root is defined by analytic continuation in p_0 from the real line to the cut plane shown in fig. 1. Eq. (9.16) implies $\mathring{p}^2 = 0$ and it follows from this that all solutions are of the form

$$ap_0 = \pm iaE \bmod 2\pi \quad \text{or} \quad ap_0 = \pi \pm iaE \bmod 2\pi \tag{9.17}$$

with $E \geq 0$ given by

$$\sinh aE = \sqrt{a^2 \mathring{\mathbf{p}}^2}. \tag{9.18}$$

Whether these are solutions of eq. (9.16) depends on the sign of $q-1$ and the sign of the square root on the right-hand side of the equation, which is determined by the analytic continuation.

In the following we restrict attention to the region

$$|\mathrm{Im}\, ap_0| < \epsilon \tag{9.19}$$

where there are no cut singularities. The condition for the existence of a pole is then that $q < 1$ at one of the values (9.17),(9.18) of p_0. Since

$$b > 1 \quad \text{if} \quad ap_0 = \pi \pm iaE \bmod 2\pi, \tag{9.20}$$

the second type of solution does not occur and we only need to consider the case $p_0 = \pm iE$. Eqs. (9.18) and (9.19) imply that such poles are also excluded unless

$$\sqrt{a^2\mathring{\mathbf{p}}^2} < \sinh\epsilon. \tag{9.21}$$

The right-hand side of this inequality is at most 1 since $\epsilon \leq \ln(1+\sqrt{2})$. Under these conditions there are precisely 8 disjoint connected regions in momentum space where (9.21) holds. Each of them is centred around one of the points

$$\mathbf{p} = \mathbf{n}\pi/a, \qquad n_k \in \{0,1\}. \tag{9.22}$$

The sign of $q-1$ at $p_0 = \pm iE$ cannot change in these regions, because $\mathring{p}^2$ and $q-1$ cannot both be equal to zero when $|\mathrm{Im}\, ap_0| < \epsilon$ (singularities of the Dirac operator are excluded in this region). Since the sign is positive at all points (9.22) with $\mathbf{n} \neq 0$, the associated regions are free of poles.

We have thus shown that the propagator has a pole in the complex p_0–plane with $|\mathrm{Im}\, ap_0| < \epsilon$ if and only if the spatial momentum components satisfy

$$a|p_k| < \pi/2 \quad \text{and} \quad \sqrt{a^2\mathring{\mathbf{p}}^2} < \sinh\epsilon. \tag{9.23}$$

Moreover the pole position is $p_0 = \pm iE$ with $E \geq 0$ given by eq. (9.18). All other singularities of the propagator are in the region $|\mathrm{Im}\, ap_0| \geq \epsilon$.

The results on the convergence at large N, the locality properties of D and the poles of the propagator derived in this section suggest that an optimal choice of parameters is

$$a_t m_0 = am_0 = 2 - \sqrt{2}. \tag{9.24}$$

The exponents ω and ϵ are then given by

$$\omega = \epsilon = \ln(1+\sqrt{2}). \tag{9.25}$$

Moreover the radius of the ball (9.23) assumes its maximal value, $\sinh\epsilon = 1$, for this choice of parameters.

10. Large N limit of $\det \mathfrak{D}$

For t-independent gauge fields the determinant of $\mathfrak{D}$ is given by eq. (5.7). This can be rewritten in the form

$$\begin{aligned}\det \mathfrak{D} &= (1/a_t)^{d_\mathrm{F}} \det\{P_+ + (RR^\dagger)^N P_-\}(\det B_+)^N \\ &= (1/a_t)^{d_\mathrm{F}} \det\{\tfrac{1}{2}aD_N\} \det\{1 + (RR^\dagger)^N\}(\det B_+)^N. \end{aligned} \tag{10.1}$$

Using the identity

$$1 + (RR^\dagger)^N = \{1 + (RR^\dagger)^{-N}\hat{P}_+ + (RR^\dagger)^N \hat{P}_-\}\{\hat{P}_- + RR^\dagger \hat{P}_+\}^N, \tag{10.2}$$

it then follows that

$$\det a_t\mathfrak{D} = \mathrm{e}^{N\Delta S} \det\{\tfrac{1}{2}aD_N\} \det\{1 + (RR^\dagger)^{-N}\hat{P}_+ + (RR^\dagger)^N \hat{P}_-\}, \tag{10.3}$$

where the action ΔS is given by

$$\Delta S = \operatorname{Tr}\bigl\{\ln B_+ + \ln[\hat{P}_- + RR^\dagger \hat{P}_+]\bigr\} \equiv a^4 \sum_x \Delta\mathcal{L}(x). \tag{10.4}$$

An important point to note here is that the density $\Delta\mathcal{L}(x)$ is a local expression in the gauge field. The first factor on the right-hand side of eq. (10.3) may hence be interpreted as a contribution to the gauge field action in 4+1 dimensions while the other factors converge to $\det\{\tfrac{1}{2}aD\}$ with exponentially small corrections.

Appendix A

To establish eq. (2.5) we choose an arbitrary basis $\phi_1(x), \ldots, \phi_{2n}(x)$ of fermion fields in four dimensions. In this basis the operator $M(t)$ becomes a complex $2n\times 2n$ matrix and the space $\mathcal{X}$ of all histories

$$M(t), \qquad 0 < t < T, \qquad \det B(t) \neq 0, \tag{A.1}$$

is thus a complex manifold of dimension $4n^2N$ where $N = T/a_t - 1$. Evidently both sides of eq. (2.5) are single-valued analytic functions on $\mathcal{X}$.

Lemma A.1. *$\mathcal{X}$ is connected. In particular, to prove eq. (2.5) it suffices to establish the relation on any open subset of $\mathcal{X}$.*

Proof: Let $M(t)$ be any given element of $\mathcal{X}$. We then construct a continuous path

$$M_s(t), \qquad 0 \leq s \leq 1, \tag{A.2}$$

in $\mathcal{X}$ which contracts $M(t)$ to zero. Explicitly the path is given by

$$a_t M_s(t) = \begin{cases} \mathrm{e}^{i2s\theta}\,[1 + a_t M(t)] - 1 & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2(1-s)\left\{\mathrm{e}^{i\theta}\,[1 + a_t M(t)] - 1\right\} & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases} \tag{A.3}$$

where the angle θ should be chosen so as to ensure that

$$B_s(t) = 1 + \tfrac{1}{2} a_t\,[M_s(t) + \gamma_5 M_s(t)\gamma_5] \tag{A.4}$$

is invertible for all s.

Along the first half of the curve we have $B_s(t) = \mathrm{e}^{i2s\theta}B(t)$ and the roots of the characteristic polynomial of this matrix are hence simply rotated in the complex plane. Since there are a finite number of roots we can choose θ so that none of them is real at $s = \frac{1}{2}$. For $\frac{1}{2} \leq s \leq 1$ the roots then move to 1 along straight lines that do not pass through 0. In particular, $B_s(t)$ is invertible for all s. □

Lemma A.2. *There exists an open subset $\mathcal{Z}$ of $\mathcal{X}$ such that, at any point in $\mathcal{Z}$, the characteristic polynomial*

$$P(\lambda) = \det\bigl\{\mathfrak{D} - \lambda\gamma_0\bigr\} \tag{A.5}$$

has d_{F} non-degenerate roots $\lambda_1, \ldots, \lambda_{d_{\mathrm{F}}}$.

Proof: Since the coefficients of $P(\lambda)$ depend analytically on the matrix elements of $M(t)$, it suffices to show that there exists one particular choice of $M(t)$ where the roots are non-degenerate (cf. sect. XII.1 of ref. [3]).

To this end it is helpful to pass to a basis in the space of fermion fields in which γ_0 is diagonal. For all t we then take $M(t)$ to be diagonal as well. Let $\mu_1, \ldots, \mu_{d_{\mathrm{F}}}$ be the diagonal entries of the matrices $\gamma_0 M(t)$, $0 < t < T$, in an arbitrary order. We may choose them to be given by $\mu_k = \mu k$. At large μ, all eigenvalues of the matrices $\gamma_0 M(t)$ are then widely separated from each other. Since

$$\bigl\|\tfrac{1}{2}\left\{\gamma_5(\partial_t^* + \partial_t) - a_t\partial_t^*\partial_t\right\}\bigr\| \leq 2/a_t, \tag{A.6}$$

the derivative part of the operator $\gamma_0\mathfrak{D}$ may be treated as a small perturbation under these conditions and one concludes that its eigenvalues are of the form

$$\lambda_k \underset{\mu\to\infty}{=} \mu_k \left\{1+\mathrm{O}[1/(a_t\mu)]\right\}, \qquad k=1,2,\ldots,d_{\mathrm{F}}. \tag{A.7}$$

Evidently these are just the roots of the polynomial $P(\lambda)$. □

We can now complete the proof of eq. (2.5) in a few lines. According to lemma A.1 and A.2 it suffices to establish the relation in all those cases where the polynomial $P(\lambda)$ [eq. (A.5)] has only non-degenerate roots. So let us assume that $\mathfrak{D}$ is of this type. For any complex number λ we then define

$$M_\lambda(t) = M(t) - \lambda\gamma_0 \tag{A.8}$$

and denote the corresponding operators $B(t)$ and $S(t)$ by $B_\lambda(t)$ and $S_\lambda(t)$ respectively. Since $B_\lambda(t) = B(t)$ it is easy to show that $S_\lambda(t)$ is a polynomial in λ satisfying

$$P_-S_\lambda(t+a_t) = \left(-a_t^2\lambda^2\right)^{t/a_t}\gamma_0 B(t)^{-1}B(t-a_t)^{-1}\ldots B(a_t)^{-1}\gamma_0 P_- + \ldots \tag{A.9}$$

at large λ. The key observation is now that

$$\det\left\{P_+ + P_-S_\lambda(T)\right\} = 0 \quad \Leftrightarrow \quad P(\lambda) = 0. \tag{A.10}$$

Up to a normalization factor the determinant hence coincides with the polynomial. The normalization can be worked out using eq. (A.9) and after setting $\lambda = 0$ the identity (2.5) is obtained. □

Appendix B

It is advantageous to work with a chiral representation of the Dirac matrices where

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}. \tag{B.1}$$

A possible choice for the 2×2 matrices e_μ is

$$e_0 = -1, \qquad e_k = -i\sigma_k \tag{B.2}$$

($k = 1, 2, 3$, and σ_k are the Pauli matrices). It is then easy to check that

$$\gamma_\mu^\dagger = \gamma_\mu, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{B.3}$$

Furthermore, if we define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, we have

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.4}$$

In particular, $\gamma_5 = \gamma_5^\dagger$ and $\gamma_5^2 = 1$.

Appendix C

To establish eq. (9.10) we first note that

$$\mathring{p}^2 = \hat{p}^2 - \tfrac{1}{4}a^2(\hat{p}^2)^2 + \tfrac{1}{4}a^2 \sum_{\mu\neq\nu} \hat{p}_\mu^2 \hat{p}_\nu^2. \tag{C.1}$$

Taking this into account we have

$$q/b = \frac{1 + (1 - a_t m_0)^2 + \left[(1 - a_t m_0)a_t a + a_t^2\right] \mathring{p}^2 + \frac{1}{4}a_t^2 a^2 \sum_{\mu\neq\nu} \hat{p}_\mu^2 \hat{p}_\nu^2}{2(1 - a_t m_0) + a_t a \hat{p}^2}. \tag{C.2}$$

As a function of $\hat{p}_0^2$ the ratio is thus of the form

$$q/b = \frac{c_1 + c_2 \hat{p}_0^2}{c_3 + c_4 \hat{p}_0^2} \tag{C.3}$$

with positive constants $c_1, \ldots, c_4$. It follows from this that q/b assumes its minimum when $\hat{p}_0^2$ is at the boundary of its domain, i.e. when $\hat{p}_0^2 = 0$ or $\hat{p}_0^2 = 4/a^2$. The same argument applies to the other momentum components as well and after some algebra one then concludes that

$$\cosh\omega = \min_{0\leq n\leq 4} \tfrac{1}{2}(r_n + 1/r_n), \qquad r_n = 1 - a_t m_0 + 2(a_t/a)n, \tag{C.4}$$

where n denotes the number of momentum components unequal to zero. In the parameter range (4.3) we have

$$r_n \geq r_1 > 1 \quad \text{for all} \quad n \geq 1 \tag{C.5}$$

so that the minimum is attained for $n = 0$ or $n = 1$. This corresponds to the cases listed in eq. (9.10).

Appendix D

As already mentioned in sect. 9, the singularities of the free Dirac operator in momentum space arise from the zeros of the argument of the square root in eq. (9.6). To find them we first note that

$$a_t^2 \mathring{p}^2 + (q-1)^2 = (q-b)(q+b). \tag{D.1}$$

The factors on the right-hand side of this equation satisfy

$$q \pm b = 0 \quad \Leftrightarrow \quad c_{\pm} + d_{\pm} a^2 \hat{p}_0^2 = 0 \tag{D.2}$$

with coefficients $c_{\pm}$ and $d_{\pm}$ given by

$$c_+ = (2 - a_t m_0)^2 + (2 - a_t m_0 + a_t/a) a_t a \hat{\mathbf{p}}^2 + \tfrac{1}{4} a_t^2 a^2 \sum_{k \neq l} \hat{p}_k^2 \hat{p}_l^2, \tag{D.3}$$

$$d_+ = (2 - a_t m_0 + a_t/a) a_t/a + \tfrac{1}{2} a_t^2 \hat{\mathbf{p}}^2, \tag{D.4}$$

$$c_- = a^2 m_0^2 + (1 - a m_0) a^2 \hat{\mathbf{p}}^2 + \tfrac{1}{4} a^4 \sum_{k \neq l} \hat{p}_k^2 \hat{p}_l^2, \tag{D.5}$$

$$d_- = 1 - a m_0 + \tfrac{1}{2} a^2 \hat{\mathbf{p}}^2 \tag{D.6}$$

Since c_+ and d_+ are both positive, the solutions of $q + b = 0$ are of the form

$$a p_0 = \pm i v, \qquad v > 0, \qquad \cosh v = 1 + \tfrac{1}{2} c_+/d_+. \tag{D.7}$$

These points lie on the imaginary axis and $\sqrt{q+b}$ thus extends to an analytic function of ap_0 in the complex plane cut from $\pm iv$ to $\pm i\infty$.

In the case of the other equation, $q - b = 0$, the situation is slightly more complicated, because d_- can be negative if $am_0 > 1$. Accordingly there are two kinds of solutions,

$$a p_0 = \pm i v, \qquad v > 0, \qquad \cosh v = 1 + \tfrac{1}{2} c_-/d_- \quad (\text{if } d_- > 0), \tag{D.8}$$

$$a p_0 = \pi \pm i v, \qquad v > 0, \qquad \cosh v = -1 - \tfrac{1}{2} c_-/d_- \quad (\text{if } d_- < 0), \tag{D.9}$$

and there is no solution if $d_- = 0$. Depending on the sign of d_-, the factor $\sqrt{q-b}$ thus extends to an analytic function of ap_0 in the complex plane cut from $\pm iv$ to $\pm i\infty$ or from $\pi \pm iv$ to $\pi \pm i\infty$. One thus ends up with the singularity structure shown in fig. 1.

To compute the exponent ϵ we now need to minimize the v's determined through eqs. (D.7)–(D.9) over all spatial momenta. This is similar to minimizing the ratio q/b (appendix C). In particular, the minimum is attained at vanishing momentum or at the corners of the Brillouin zone. The minima are

$$\text{from eq. (D.7):} \quad \mathrm{e}^{\epsilon} = 1 - am_0 + 2a/a_t, \tag{D.10}$$

$$\text{from eq. (D.8):} \quad \mathrm{e}^{\epsilon} = \begin{cases} (1-am_0)^{-1} & \text{if } am_0 \leq 2-\sqrt{2}, \\ 3 - am_0 & \text{otherwise,} \end{cases} \tag{D.11}$$

$$\text{from eq. (D.9):} \quad \mathrm{e}^{\epsilon} = (am_0 - 1)^{-1} \quad \text{if} \quad am_0 > 1. \tag{D.12}$$

Taking the minimum of these three values yields the result quoted in sect. 9.

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