The gauge anomaly: algebraic & topological facts

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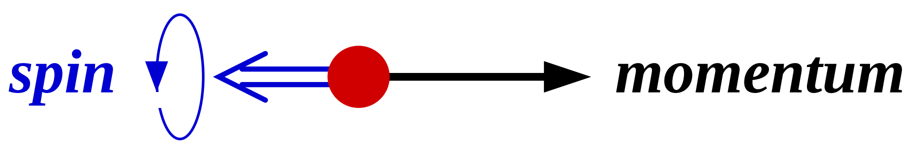
Neutrinos produced through $e + p \rightarrow n + \nu_e$ are left–handed



i.e. they are chiral

Infinitesimal chiral transformations

$$\psi \rightarrow \psi + \epsilon\delta\psi, \qquad \delta\psi \equiv \gamma_5\psi$$

preserve the (massless) Dirac equation

$$D\psi = 0, \qquad D \equiv \gamma_\mu \partial_\mu$$

Left-handed spin $\frac{1}{2}$ particles are described by fields satisfying

$$\gamma_5\psi = -\psi$$

i.e.

$$P_+\psi = 0, \quad P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5)$$

Two-component formulation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ “chiral” representation

$$\gamma_5 \psi = -\psi, \quad D\psi = 0 \quad \Leftrightarrow \quad \psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \qquad \sigma_\mu \partial_\mu \chi = 0$$

Also referred to as Weyl fermions

Coupling to gauge fields

$$\psi_{A\alpha}, \quad A = 1, \ldots, 4 \quad \text{Dirac index}$$

$$\alpha = 1, \ldots, N \quad \text{“flavour” index}$$

Flavour symmetry group G

$$\psi \to R(\Lambda)\psi, \quad \Lambda \in G$$

Gauge field & gauge-covariant differential operators

$$A_\mu = A_\mu^a T^a, \qquad D_\mu \psi = \left\{ \partial_\mu + A_\mu^a R(T^a) \right\} \psi$$

"Pure" chiral gauge theories

Euclidean action

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \operatorname{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \overline{\psi}(x)\gamma_\mu D_\mu \psi(x) \right\}$$

where

$$P_+\psi = 0, \quad \overline{\psi}P_- = 0, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

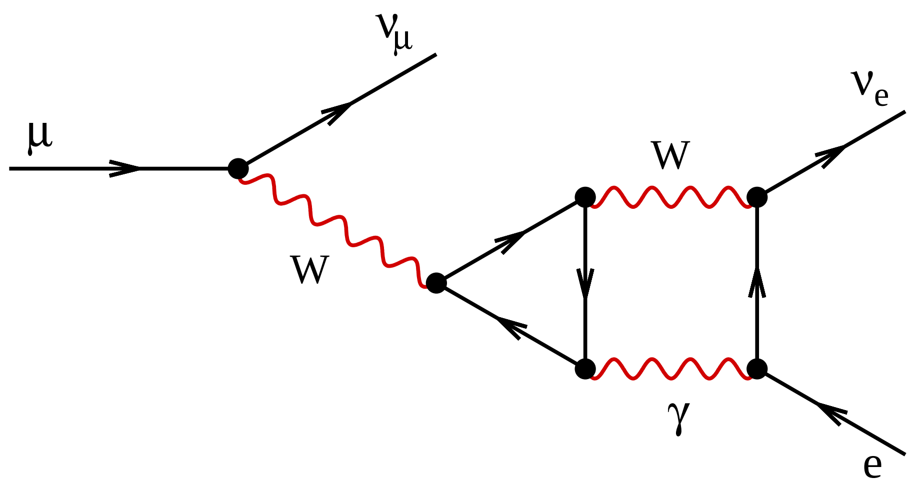
- Form dictated by gauge invariance & power counting
- Charge conjugation maps right- to left-handed fermions

At the quantum level, these theories appear to be rather artificial

- ⋆ gauge anomaly
- ⋆ global topological obstructions
- ⋆ difficult to put on a lattice

Actually inconsistent unless G and R satisfy certain conditions

$\Rightarrow$ perturbative calculations are difficult



and we know nearly nothing about the non-perturbative properties of these theories

Fundamental issues

Are chiral gauge theories consistent beyond perturbation theory ?

Can they be put on a lattice (or be regularized otherwise) without breaking the gauge symmetry ?

Is there a natural way in which chiral fermions can arise ?

A little history, necessarily incomplete

1981	Nielsen–Ninomiya theorem	Nielsen, Ninomiya Friedan
1982	Ginsparg–Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$	Ginsparg, Wilson
1983	Chiral fermions from 4+1 dimensions Relation to descent equations	Rubakov, Shaposhnikov Callan, Harvey
1992	Lattice domain-wall fermions	D. Kaplan Shamir, Furman
1998	Perfect lattice Dirac operator Neuberger–Dirac operator Exact index theorem & chiral symmetry	Hasenfratz Neuberger Hasenfratz, Niedermayer, Laliena M.L.
1999	Chiral lattice gauge theories Local cohomology, global anomalies	M.L. H. Suzuki Kikukawa Bär, Campos

Topics covered in the lectures

- ⋆ Algebraic & topological aspects of the gauge anomaly — an old subject!
- ⋆ Local cohomology on the lattice
- ⋆ Lattice fermions & the Ginsparg–Wilson relation
- ⋆ Chiral lattice gauge theories

M.L., Erice Lectures 2000, hep-th/0102028

Gauge anomaly

Effective action in a classical background field

$$\mathrm{e}^{-S_{\mathrm{eff}}[A]} = \int \mathrm{D}[\psi]_L \mathrm{D}[\overline{\psi}]_L \, \mathrm{e}^{-\int \mathrm{d}^4 x \, \overline{\psi} \gamma_\mu D_\mu \psi}$$

Induced gauge current

$$j_\mu^a \equiv \frac{\delta S_{\mathrm{eff}}}{\delta A_\mu^a} = \langle \overline{\psi} \gamma_\mu R(T^a) \psi \rangle$$

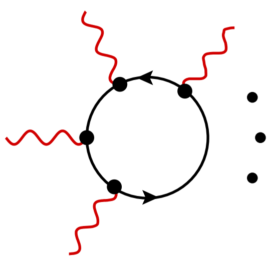
$$\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega] \equiv D_\mu \omega \quad \Rightarrow \quad \delta S_{\mathrm{eff}} = \int \mathrm{d}^4 x \, D_\mu \omega^a(x) j_\mu^a(x)$$

S_{eff} is gauge-invariant $\Leftrightarrow$ j_μ is gauge-covariant and $D_\mu j_\mu = 0$

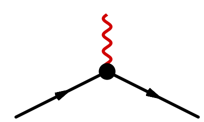
S_{eff} generates the fermion one-loop diagrams

$$S_{\text{eff}} = -\sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 p_n}{(2\pi)^4} (2\pi)^4 \delta(p_1 + \ldots + p_n)$$

$$\times V^{(n)}(p_1, \ldots, p_n)^{a_1 \ldots a_n}_{\mu_1 \ldots \mu_n} \widetilde{A}^{a_1}_{\mu_1}(p_1) \ldots \widetilde{A}^{a_n}_{\mu_n}(p_n)$$

$V^{(n)} =$  $+ \ldots$

 $= -i \dfrac{\gamma_\mu p_\mu}{p^2} P_+$

 $= \gamma_\mu R(T^a)$

Power counting implies that S_{eff} is well-defined up to

$$S_{\text{eff}} \to S_{\text{eff}} + \int \mathrm{d}^4x\, \Omega(x)$$

$$\uparrow$$

local polynomial in $A_\mu, \partial_\mu A_\nu, \ldots$ of dimension ≤ 4

To compute S_{eff} one can use a momentum cutoff

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 + \Lambda^2},$$

for example, or a proper time representation

H. Leutwyler, Phys. Lett. B152 (1985) 78

A detailed calculation yields

$$\delta S_{\text{eff}} = \frac{i}{192\pi^2} \int \mathrm{d}^4x \, \epsilon_{\mu\nu\rho\sigma} d_R^{abc} \omega^a \left\{ \partial_\mu A_\nu^b \partial_\rho A_\sigma^c + \tfrac{1}{2} \partial_\mu \left(A_\nu^b F_{\rho\sigma}^c \right) \right\}$$

$$+ \int \mathrm{d}^4x \, \delta\Omega(x)$$

$$\boxed{d_R^{abc} \equiv 2i \, \mathrm{tr} \left\{ R(T^a) R(T^b) R(T^c) + (b \leftrightarrow c) \right\}}$$

Gauge invariance can be preserved $\Leftrightarrow d_R^{abc} = 0$

Anomaly-free representations

Gauge group $\mathrm{U}(1)$

There is only one generator and

$$R(T^1) = i \times \mathsf{diag}(\mathrm{e}_1, \ldots, \mathrm{e}_N), \qquad d_R^{111} = 4 \sum_{\alpha=1}^{N} \mathrm{e}_\alpha^3$$

For example, $\mathrm{e}_1 = \ldots = \mathrm{e}_8 = 1,\ \mathrm{e}_9 = -2$, is anomaly-free

Real & pseudo-real representations

These are all anomaly-free

$G = \mathrm{SU}(2)$ has only such representations and is therefore "safe"

Gauge group $\mathrm{SU}(n),\ n \geq 3$

For any representation R we have

$$d_R^{abc} = c_R \times d^{abc}$$

$\uparrow$

d–symbol in the fundamental representation

To compute c_R it suffices to consider a single generator

$$T = i \times \mathrm{diag}(1, \ldots, 1, 1-n), \qquad c_R = \mathrm{tr}\{R(T)^3\}/\mathrm{tr}\{T^3\}$$

Example:

The fermions in the standard SU(5) GUT are in the anomaly-free representation $R = \mathbf{5}^* \oplus \mathbf{10}$ Georgi & Glashow 1974

Topological interpretation

Recall

$$\delta S_{\text{eff}} = \frac{i}{192\pi^2} \int \mathrm{d}^4x \, \epsilon_{\mu\nu\rho\sigma} d_R^{abc} \omega^a \left\{ \partial_\mu A_\nu^b \partial_\rho A_\sigma^c + \tfrac{1}{2} \partial_\mu \left(A_\nu^b F_{\rho\sigma}^c \right) \right\}$$

This expression is reminiscent of the second Chern character

$$\mathrm{ch}_2 \propto \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$$

but

- the algebraic structure is different
- and it depends on $\omega \Rightarrow$ the anomaly is a local obstruction

In the following consider

- ⋆ gauge group $G = \mathrm{SU}(n)$, $n \geq 3$
- ⋆ fundamental representation R

All other simple groups are actually safe

Descent equations

Stora '76, Zumino '83

$d = 6:$ Chern character

$\downarrow$ $\qquad$ $\mathrm{tr}\{F^3\} = \mathrm{d}Q_5^0$

$d = 5:$ Chern-Simons term

$\downarrow$ $\qquad$ $\delta_\omega Q_5^0 = \mathrm{d}Q_4^1$

$d = 4:$ gauge anomaly

$A = A_\mu \mathrm{d}x_\mu, \quad F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x_\mu \mathrm{d}x_\nu$ gauge potential, field strength

$\delta_\omega A = \mathrm{d}\omega + [A, \omega]$ gauge variation

$Q_5^0 = \mathrm{tr}\{A(\mathrm{d}A)^2 + \frac{3}{2}A^3\mathrm{d}A + \frac{3}{5}A^5\}$ Chern-Simons term

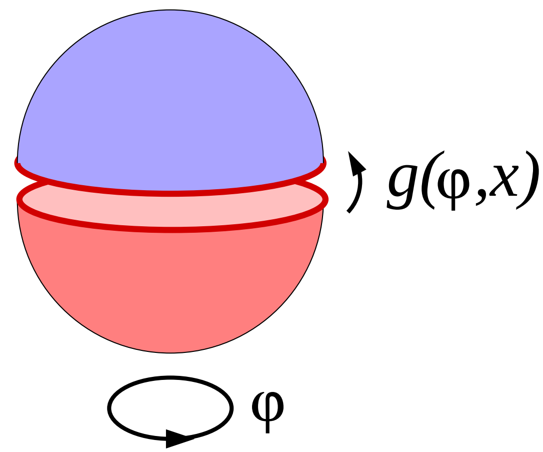
$Q_4^1 = \mathrm{tr}\{\omega\mathrm{d}[A\mathrm{d}A + \frac{1}{2}A^3]\}$ gauge anomaly

Gauge anomaly $\Leftrightarrow$ *non-trivial bundles on* $S^2 \times S^4$

Alvarez-Gaumé & Ginsparg '84

- $g(\varphi, x) : S^1 \times S^4 \rightarrow \mathrm{SU}(n)$ defines a principal bundle on $S^2 \times S^4$

- This bundle can be non-trivial since $\pi_5(\mathrm{SU}(n)) = \mathbb{Z}$



$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}^2 \quad \text{(in 6 dimensions)}$$

$$\int_{S^2 \times S^4} \mathrm{ch}_3 = -\frac{i}{48\pi^3} \int_{S^2 \times S^4} \mathrm{tr}\{\mathcal{F}^3\} \in \mathbb{Z}$$

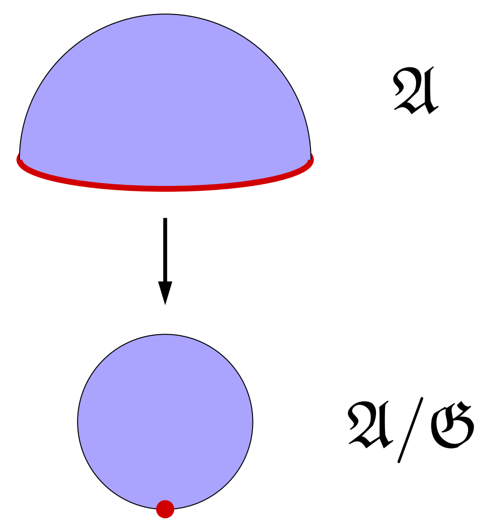
The closed gauge curve

$$A^g = g^{-1}Ag + g^{-1}\mathrm{d}g, \quad 0 \leq \varphi \leq 2\pi$$

defines a gauge connection on the equator of $S^2 \times S^4$

Choosing any extension of A^g to the bundle one finds

$$\Delta S_{\mathrm{eff}} \equiv \int_0^{2\pi} \mathrm{d}\varphi \, \frac{\partial S_{\mathrm{eff}}}{\partial \varphi} = 2\pi i \int_{S^2 \times S^4} \mathrm{ch}_3$$



- $\mathrm{Im}\, S_{\mathrm{eff}}$ is multi-valued
- $\Rightarrow$ anomaly cannot be removed
- Topological obstruction $\Leftrightarrow \pi_2(\mathfrak{A}/\mathfrak{G}) \neq 0$

Summary

- ⋆ Gauge anomaly $\leftrightarrow \mathrm{ch}_3$
- ⋆ Topology of field space matters
- ⋆ Quantization must take this into account