Local cohomology in lattice gauge theory

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Gauge anomaly $\Leftrightarrow$ *third Chern class* $\Leftrightarrow$ *topology of field space*

Andrejewski Lectures II, Berlin, February 2004

Topological fields

Pure gauge theory on $\mathbb{R}^n$, gauge group G

A gauge-invariant local field $q(x)$ is called <u>topological</u> if

$$\int \mathrm{d}^n x \; \delta q(x) = 0$$

for all variations $\delta A^a_\mu(x)$ with compact support

A trivial case is

$$q = \partial_\mu k_\mu, \qquad k_\mu : \text{local, gauge-invariant}$$

The Chern monomials

$$q = c_{\mu_1 \ldots \mu_{2m}} t^{a_1 \ldots a_m} F^{a_1}_{\mu_1 \mu_2} \ldots F^{a_m}_{\mu_{2m-1} \mu_{2m}}$$

$c_{\mu_1 \ldots \mu_{2m}}$: totally anti-symmetric

$t^{a_1 \ldots a_m}$: G-invariant, totally symmetric

are examples of non-trivial topological fields

Cohomology problem:

"Find a complete basis of topological fields q modulo $\partial_\mu k_\mu$ terms"

In the continuum theory we have

Theorem

Any topological field q that is a polynomial in the gauge potential and its derivatives is of the form

$$q = c + \partial_\mu k_\mu$$

where c is a Chern polynomial and k_μ a gauge-invariant local current

Brandt, Dragon & Kreuzer '89

Dubois-Violette et al. '91

- Can be shown using the descent equations
- or, more directly, the Poincaré lemma

Reduction to the abelian case

Define $\mathring{q}$ through

$$A^a_\mu \to t A^a_\mu \quad \Rightarrow \quad q = t^\nu \mathring{q} + \mathrm{O}(t^{\nu+1})$$

⋆ $\mathring{q}$ is a homogeneous polynomial of degree ν

⋆ that is invariant under

$$A_\mu \to g A_\mu g^{-1} \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu \omega$$

($=$ linearized gauge transformations)

⋆ and which is topological, i.e. $\int \mathrm{d}^n x \, \delta \mathring{q} = 0$

It suffices to show that any such field is of the form $\mathring{c} + \partial_\mu \mathring{k}_\mu$

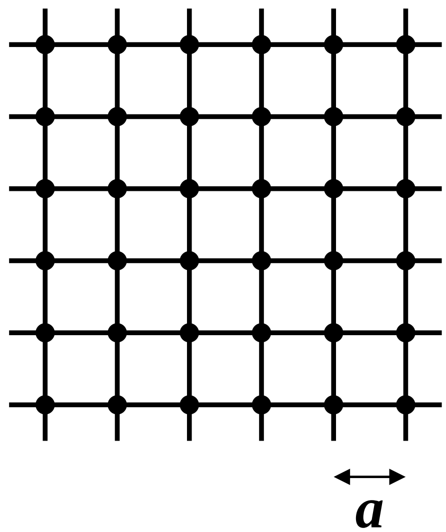
The abelian case may be solved using the fact that

$$\mathrm{d}f = 0 \quad \Rightarrow \quad f = \mathrm{d}g + \underset{\substack{\uparrow \\ \text{constant}}}{c}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$$

for differential forms on $\mathbb{R}^n$, which is the classical Poincaré lemma

Lattice gauge theory (mini-intro)

Replace space-time by a 4–dimensional hypercubic lattice



Fermion field

$$\psi(x), \quad x = a\,(n_0, n_1, n_3, n_4)\,, \quad n_\mu \in \mathbb{Z}$$

$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{ipx}\widetilde{\psi}(p)$$

$\Rightarrow$ momentum cutoff $|p_\mu| \leq \pi/a$

Difference operators

$$\partial_\mu \psi(x) = \{\psi(x + a\hat{\mu}) - \psi(x)\} / a$$

$$\partial_\mu^* \psi(x) = \{\psi(x) - \psi(x - a\hat{\mu})\} / a$$

$\uparrow$ unit vector in direction μ

Wilson–Dirac operator

$$D_{\mathrm{w}} = \tfrac{1}{2}\{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu\}$$

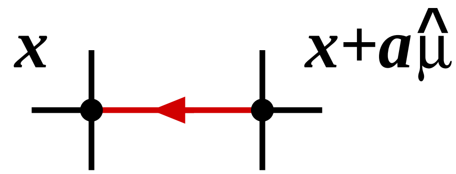
$$= i\gamma_\mu \mathring{p}_\mu + \tfrac{1}{2}a\hat{p}^2 \qquad \text{(in momentum space)}$$

$$\mathring{p}_\mu \equiv (1/a)\sin(ap_\mu), \qquad \hat{p}_\mu \equiv (2/a)\sin(ap_\mu/2)$$

Gauge-covariant difference operator

$$\nabla_\mu \psi(x) = \{ R[U(x,\mu)]\psi(x+a\hat{\mu}) - \psi(x) \} / a$$

$U(x,\mu) \in G$ (the lattice gauge field)



$$U(x,\mu) \to \Lambda(x) U(x,\mu) \Lambda(x+a\hat{\mu})^{-1}$$

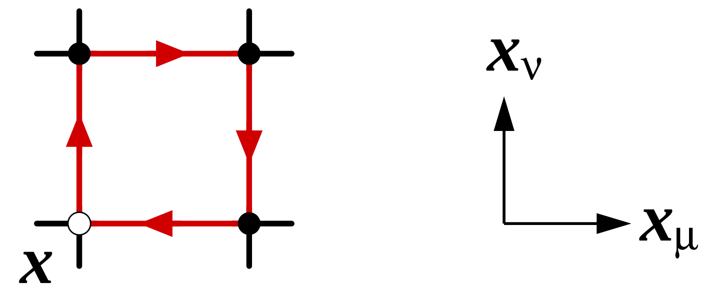
In the classical continuum limit

$$U(x,\mu) = 1 + aA_\mu(x) + \mathrm{O}(a^2)$$

$$\Rightarrow \quad \nabla_\mu = D_\mu + \mathrm{O}(a)$$

Discretization of the Yang–Mills action

$$U_{\Box} = 1 + a^2 F_{\mu\nu}(x) + \mathrm{O}(a^3)$$



$$S = \frac{1}{g^2} \sum_{\Box} \mathrm{tr}\{(U_{\Box} - 1)^{\dagger}(U_{\Box} - 1)\} = \frac{2}{g^2} \sum_{\Box} \mathrm{Re}\,\mathrm{tr}\{1 - U_{\Box}\}$$

Euclidean correlation functions (Wilson loops etc.)

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int_{\mathrm{fields}} \mathrm{D}[U]\, \mathcal{O}[U]\, \mathrm{e}^{-S[U]}, \qquad \mathrm{D}[U] \equiv \prod_{x,\mu} \mathrm{d}U(x,\mu)$$

$\uparrow$ G-invariant measure

Wilson '74

Local fields $\phi(x)$ on the lattice are

- $\star$ smooth functions of the field variables at $y = x + \mathrm{O}(a)$
- $\star$ that transform like a scalar field under translations

such as

$$\mathrm{Re}\,\mathrm{tr}\{1 - U_{\Box}\}, \quad \overline{\psi}\sigma_{\mu\nu}\nabla_{\mu}\nabla_{\nu}\psi, \quad \overline{\psi}T^{a}\gamma_{\mu}\psi, \quad \ldots$$

[exponential localization with a range of $\mathrm{O}(a)$ may be allowed]

Topology in lattice gauge theory

The absence of continuity in space implies

- lattice gauge fields are homotopic to $U = 1$
- there are no non-trivial topological fields

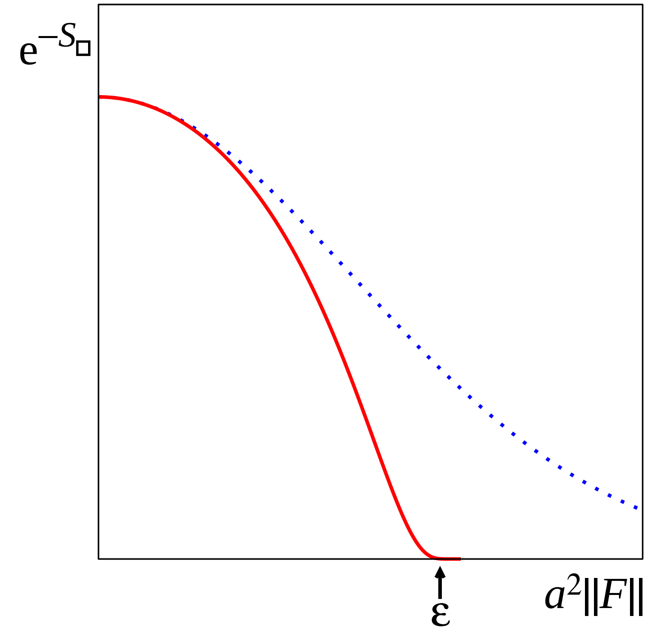
Topology is recovered if

$$a^2\|F_{\mu\nu}\| \equiv \|1 - U_\Box\| \leq \epsilon$$

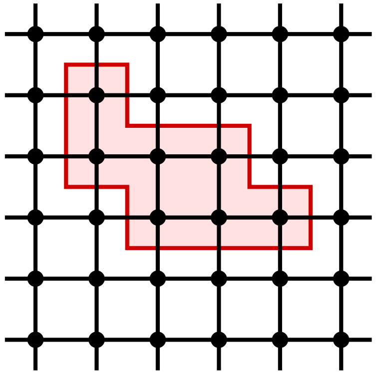
for some fixed sufficiently small ϵ

Can be imposed using a modified action

M.L. '98, Fukaya & Onogi '03



A gauge-invariant local field q is topological if



$$\sum_x \delta q(x) = 0$$

for all variations $\delta U(x,\mu)$ of the gauge field with bounded support

Note: it suffices to define q for all fields U satisfying $\|1 - U_{\Box}\| \leq \epsilon$

Cohomology problem

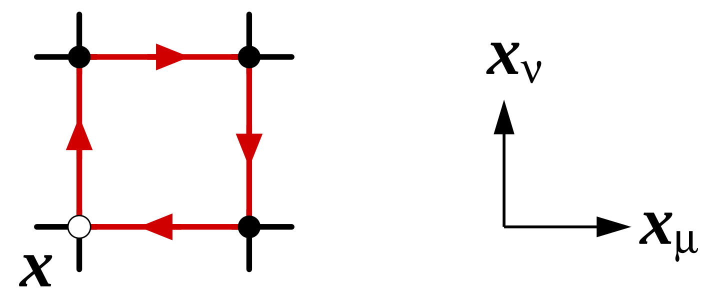
Find all topological fields up to derivative terms $\partial_\mu^* k_\mu$

(where k_μ is any gauge-invariant local current)

U(1) theory

Field tensor

$$U_{\Box} = \mathrm{e}^{ia^2 F_{\mu\nu}}, \qquad |a^2 F_{\mu\nu}| < \pi$$



This is a gauge-invariant smooth local field since $U_{\Box} = -1$ is excluded by the constraint $|1 - U_{\Box}| \leq \epsilon$

The general Chern polynomial

$$c(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x + a\hat{\mu} + a\hat{\nu}) + \ldots$$

is an example of a topological field

Assume now that $\epsilon < \frac{1}{3}\pi$. Then we have

Theorem

Any topological field q is of the form

$$q = c + \partial_\mu^* k_\mu$$

where c is a Chern polynomial and k_μ a gauge-invariant local current

M.L. '98, Fujiwara, Suzuki, Wu '99, Suzuki '00

Proof follows the one in the continuum theory, with some complications (field space, Leibniz rule)

Sketch of the proof

2d lattice, setting $a = 1$ and $U(x,\mu) = \mathrm{e}^{iA_\mu(x)}$ for simplicity

$$q(x) = \alpha + \sum_y \underbrace{\int_0^1 \mathrm{d}t \left(\frac{\partial q(x)}{\partial A_\mu(y)} \right)_{A \to tA}}_{j_\mu(x,y)} A_\mu(y)$$

The following properties hold

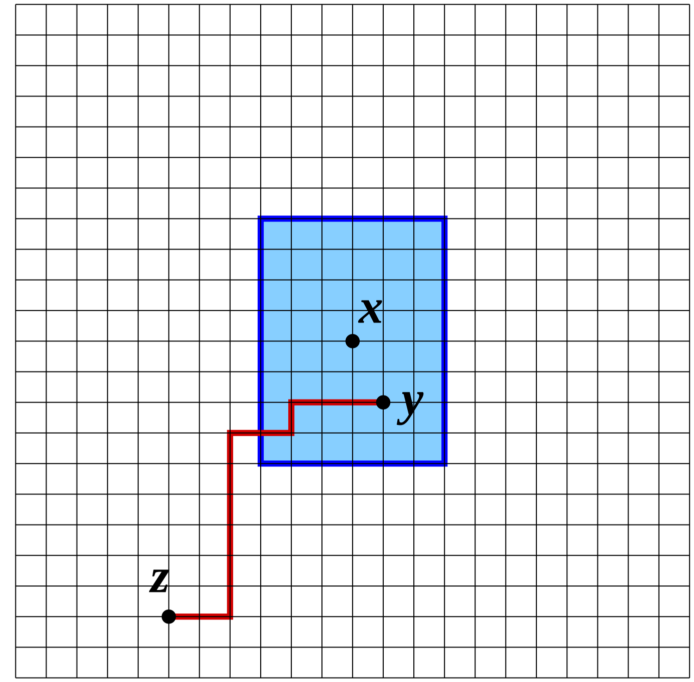
- ⋆ $j_\mu(x,y)$ is local & gauge-invariant
- ⋆ $\sum_x j_\mu(x,y) = 0$ [since q is topological]
- ⋆ $j_\mu(x,y)\overleftarrow{\partial}_\mu^* = 0$ [since q is gauge-invariant]

The last equation implies

$$j_\mu(x,y) = \phi(x,y)\overleftarrow{\partial}^*_\nu \epsilon_{\mu\nu}$$

where

- $\phi(x,y)$ is local & gauge invariant
- $q(x) = \alpha + \frac{1}{2}\sum_y \phi(x,y)\epsilon_{\mu\nu}F_{\mu\nu}(y)$
- $\sum_x \phi(x,y) =$ constant $\equiv 2\beta$



So if we set $\theta(x,y) \equiv \phi(x,y) - 2\beta\delta_{xy}$ we have $\sum_x \theta(x,y) = 0$ and

$$q(x) = \alpha + \beta\epsilon_{\mu\nu}F_{\mu\nu}(x) + \frac{1}{2}\sum_y \theta(x,y)\epsilon_{\mu\nu}F_{\mu\nu}(y)$$

The last term is a divergence term since

$$\theta(x,y) = \theta_0(x,y) + \theta_1(x,y), \qquad \theta_1(x,y) \equiv \delta_{x_0 y_0} \textstyle\sum_{z_0} \theta(z,y)|_{z_1=x_1}$$

$$\textstyle\sum_{x_0} \theta_0(x,y) = \sum_{x_1} \theta_1(x,y) = 0$$

$$\Rightarrow \quad \theta_0(x,y) = \partial_0^* h_0(x,y), \qquad \theta_1(x,y) = \partial_1^* h_1(x,y)$$

We have thus shown that

$$q(x) = \alpha + \beta \epsilon_{\mu\nu} F_{\mu\nu}(x) + \partial_\mu^* k_\mu(x)$$

$$k_\mu(x) = \tfrac{1}{2} \sum_y h_\mu(x,y) \epsilon_{\nu\rho} F_{\nu\rho}(y)$$

QED

SU(n) gauge theories

The cohomology problem is much harder in this case!

Reduction to the abelian case?

$$\mathrm{SU}(n) \rightarrow \mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \qquad \text{(diagonal subgroup)}$$

$q = c + \partial_\mu^* k_\mu$ on the subspace of all these fields

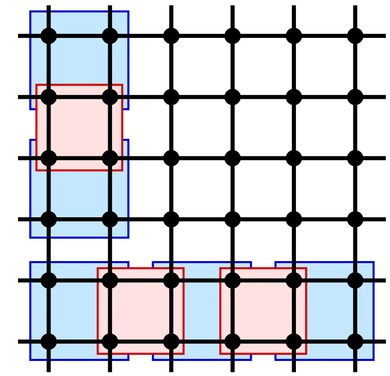
Presumably the cohomology classes are labelled by the associated abelian Chern monomials c

Suzuki '00, M.L. '00, Igarashi, Okuyama & Suzuki '02

Explicit constructions of topological fields

★ Geometric constructions

- Principal bundles $\leftrightarrow$ cohomology classes
- Characterized by transition functions
- Obtain these using smooth interpolation



M.L. '82, Phillips & Stone '86,'90

★ Via the axial anomaly & the Ginsparg–Wilson relation

Hasenfratz, Laliena & Niedermayer '98