Lattice fermions & the Ginsparg–Wilson relation

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Topics

- ⋆ The Nielsen–Ninomiya theorem
- ⋆ Exact chiral symmetry recovered
- ⋆ Locality $\leftrightarrow$ topology $\leftrightarrow$ chirality

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Free lattice fermion fields

$$\psi(x), \quad x/a \in \mathbb{Z}^4$$

$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{ipx} \widetilde{\psi}(p)$$

Massless lattice Dirac operator

$$D = \tfrac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu) + \mathrm{O}(a)$$

$$= i\gamma_\mu p_\mu + \mathrm{O}(a) \qquad \text{(in momentum space)}$$

Euclidean action

$$S = a^4 \sum_x \overline{\psi}(x) D \psi(x)$$

By minimal substitution

$$\partial_\mu \rightarrow \nabla_\mu, \qquad \partial_\mu^* \rightarrow \nabla_\mu^*,$$

any lattice Dirac operator of this type becomes gauge-covariant

This operation preserves chirality $\{\gamma_5, D\} = 0$

$\Rightarrow$ gauge-invariant lattice regularization of chiral gauge theories

However:

"Any consistent regularization of these theories that preserves the gauge symmetry must refer to the fermion representation R"

The Nielsen–Ninomiya theorem

Any (free) lattice Dirac operator D is of the form

$$[D\psi](x) = a^4 \sum_y D(x-y)\psi(y), \qquad \widetilde{D}(p) = a^4 \sum_z \mathrm{e}^{-ipz} D(z)$$

Desirable properties:

(1) $\widetilde{D}(p)$ is a smooth function of p_μ with period $2\pi/a$

(2) For $p \ll \pi/a$ we have $\widetilde{D}(p) = i\gamma_\mu p_\mu + \mathrm{O}(ap^2)$

(3) $\widetilde{D}(p)$ is invertible at all $p \neq 0$ (mod $2\pi/a$)

(4) $\{\gamma_5, \widetilde{D}(p)\} = 0$

Theorem: (1)–(4) are incompatible**!** (Nielsen & Ninomiya '82)

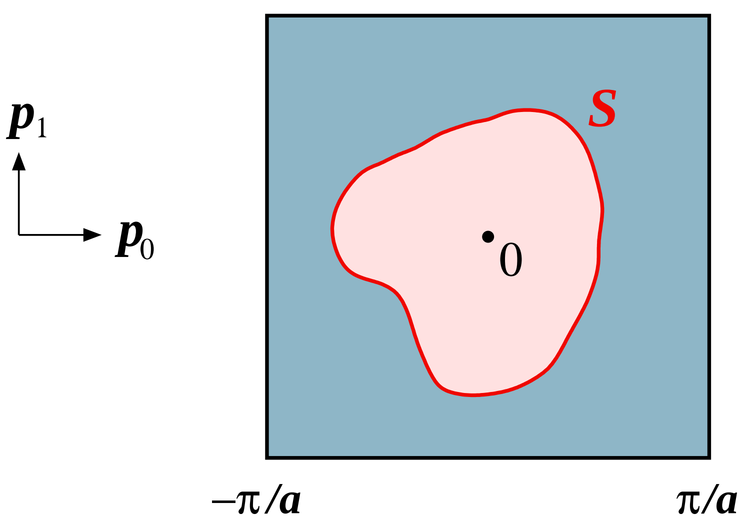
Proof (Friedan '82)

In a chiral representation of the γ matrices

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{D} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Topological current

$$j_\mu \equiv \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\big\{(B\partial_\nu B^{-1})(B\partial_\rho B^{-1})(B\partial_\sigma B^{-1})\big\}$$ (derivatives wrt p_μ)



$\int_S \mathrm{d}\sigma_\mu \, j_\mu$ is independent of S

$\Rightarrow 1 = 0$!

Wilson–Dirac operator (Wilson '74)

$$D_{\mathrm{w}} = \tfrac{1}{2}\{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu\}$$

$$= i\gamma_\mu \mathring{p}_\mu + \tfrac{1}{2}a\hat{p}^2 \qquad \text{(in momentum space)}$$

$$\mathring{p}_\mu \equiv (1/a)\sin(ap_\mu), \qquad \hat{p}_\mu \equiv (2/a)\sin(ap_\mu/2)$$

This operator satisfies (1)–(3) but violates chirality at $\mathrm{O}(a)$

- Not a fundamental difficulty in lattice QCD
- But a serious obstacle for chiral lattice gauge theories

Ginsparg–Wilson relation

Instead of $\{\gamma_5, D\} = 0$ require

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

Ginsparg & Wilson '82

If this relation holds, and also properties (1)-(3), we have

- ⋆ Exact chiral symmetry on the lattice
- ⋆ Consistent chiral projectors
- ⋆ Chiral zero modes and an exact index theorem

Exact chiral symmetry

The fermion action

$$S = a^4 \sum_x \overline{\psi}(x) D \psi(x)$$

is invariant under the infinitesimal transformation

$$\psi \to \psi + \epsilon \gamma_5 \left(1 - aD\right) \psi, \qquad \overline{\psi} \to \overline{\psi} + \epsilon \overline{\psi} \gamma_5$$

A flavour matrix may be included if the fields carry a flavour index

M.L. '98

Chiral projectors

The operator $\hat{\gamma}_5 \equiv \gamma_5(1 - aD)$ satisfies

$$\gamma_5 D = -D\hat{\gamma}_5$$

$$(\hat{\gamma}_5)^2 = 1, \qquad (\hat{\gamma}_5)^\dagger = \hat{\gamma}_5 \qquad \text{(assuming } D^\dagger = \gamma_5 D \gamma_5\text{)}$$

Chiral projectors for *fermion* and *antifermion* fields

$$\hat{P}_\pm = \tfrac{1}{2}(1 \pm \hat{\gamma}_5), \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

In particular, the zero modes of D can be taken to be chiral, viz.

$$D\psi = 0, \quad \gamma_5\psi = \pm\psi$$

Index theorem

On a finite lattice with periodic b.c. we have

$$\text{index}(D) \equiv n_+ - n_- = \mathrm{Tr}\left\{\gamma_5\left(1 - \tfrac{1}{2}aD\right)\right\} \qquad (*)$$

Moreover, as we shall see later, the rhs is of the form

$$a^4 \sum_x q(x)$$

where q is a local topological field representing ch_2

Hasenfratz, Laliena & Niedermayer '98

Proof of $(*)$: $\mathrm{Tr}\left\{\gamma_5\left(1 - \tfrac{1}{2}aD\right)P_\lambda\right\} = \left(1 - \tfrac{1}{2}a\lambda\right)\mathrm{Tr}\left\{\gamma_5 P_\lambda\right\}$

$$= \left(1 - \tfrac{1}{2}a\lambda^*\right)\mathrm{Tr}\left\{\gamma_5 P_\lambda\right\} = 0 \text{ if } \lambda \neq 0$$

Lattice Dirac operators satisfying the GW relation

$\star$ Block-spin RG construction, "perfect" action approach

Ginsparg & Wilson '82, Hasenfratz '97, Hasenfratz, Laliena & Niedermayer '98

$\star$ Domain-wall fermions, reduction to 4d

Kaplan '92, Neuberger '98, ...

$$A = 1 - aD_{\mathrm{w}}, \qquad D_{\mathrm{w}}\text{: Wilson–Dirac operator}$$

$$D \equiv \frac{1}{a}\left\{1 - A\left(A^{\dagger}A\right)^{-1/2}\right\} = \tfrac{1}{2}\left(D_{\mathrm{w}} - D_{\mathrm{w}}^{\dagger}\right) + \mathrm{O}(a)$$

(the GW relation actually holds for any A)

Are there unphysical poles?

In momentum space

$$a\widetilde{D} = 1 - \frac{1 - \frac{1}{2}a^2\hat{p}^2 - ia\gamma_\mu \mathring{p}_\mu}{\sqrt{1 + \frac{1}{2}a^4 \sum_{\mu<\nu} \hat{p}_\mu^2 \hat{p}_\nu^2}}$$

$\Rightarrow$ the propagator has no poles at $p \neq 0$ and $\widetilde{D} = i\gamma_\mu p_\mu + \mathrm{O}(ap^2)$

Locality?

D is local in the sense that

$$D\psi(x) = a^4 \sum_y D(x,y)\psi(y)$$

$$\|D(x,y)\| \leq C\mathrm{e}^{-\|x-y\|/\varrho}, \qquad \varrho \sim \text{a few lattice spacings}$$

Unitarity?

The propagator admits a Källén-Lehmann representation

$$\langle \psi(x)\overline{\psi}(y)\rangle \underset{x_0>y_0}{=} \int_0^\infty \mathrm{d}E \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \varrho(E,\mathbf{p})\mathrm{e}^{-E(x_0-y_0)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}$$

such that the spectral density

$$\zeta^\dagger \gamma_0 \varrho(E,\mathbf{p})\zeta$$

is non-negative for any complex Dirac spinors ζ

$\Rightarrow$ the free theory is unitary

Including the gauge field

Use the same formula

$$A = 1 - aD_{\mathrm{w}}, \qquad D_{\mathrm{w}}\text{: Wilson–Dirac operator}$$

$$D \equiv \frac{1}{a}\left\{1 - A\left(A^{\dagger}A\right)^{-1/2}\right\}$$

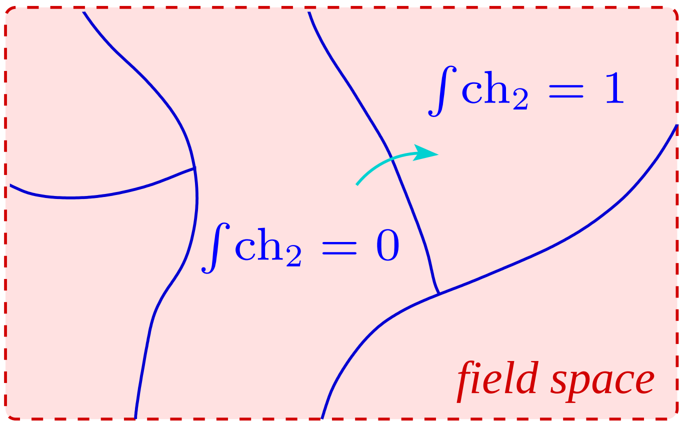
but now include the gauge field U in D_{w}

- D is gauge-covariant
- GW relation satisfied $\Rightarrow$ exact chiral symmetry, ...
- However: the inverse square root may be singular at some U

Actually, since

$$\text{index}(D) = \mathrm{Tr}\left\{\gamma_5\left(1-\tfrac{1}{2}aD\right)\right\} = -a^4\sum_x \tfrac{1}{2}a\,\mathrm{tr}\left\{\gamma_5 D(x,x)\right\}$$

the position space kernel $D(x,y)$ <u>must</u> be discontinuous



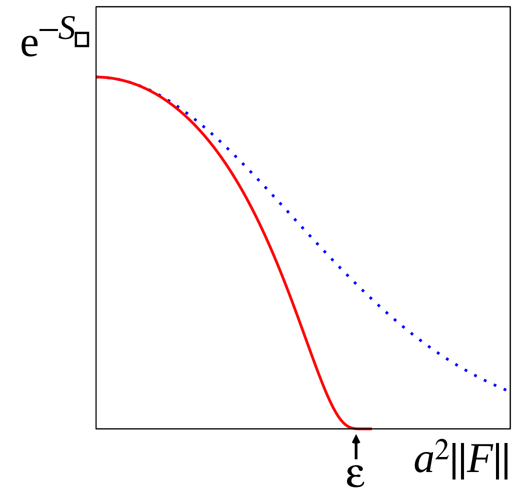
D may also become <u>non-local</u> near the sector boundaries

Theorem

On the subspace of gauge fields satisfying

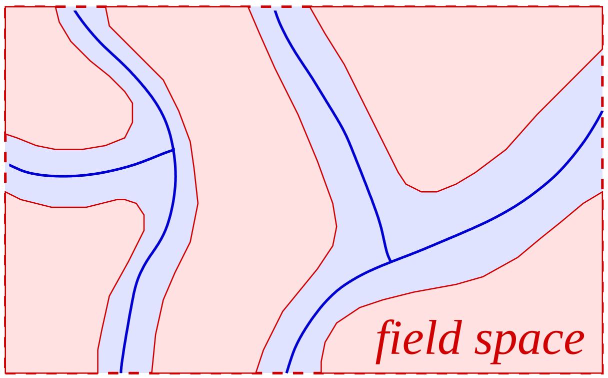
$$a^2\|F_{\mu\nu}\| \equiv \|1 - U_{\Box}\| \leq \epsilon < \tfrac{1}{30}$$

we have



1. $\|D(x,y)\| \leq C\,\mathrm{e}^{-\|x-y\|/\varrho}$

2. $D(x,y)$ is a smooth local function of the field variables $U(z,\nu)$



where $\varrho = \mathrm{O}(a)$ *depends on* ϵ

Hernández, Jansen & M.L. '98

Chern character

In the continuum theory in $d = 2n$ dimensions

$$\text{index}(\not{D}) = -\int \text{ch}_n$$

$$\text{ch}_n = \frac{i^n}{(2\pi)^n n!} \operatorname{tr}\{F^n\}, \qquad F = \tfrac{1}{2} F_{\mu\nu} \mathrm{d}x_\mu \mathrm{d}x_\nu$$

(assuming the fermions are in the fundamental representation)

On the lattice, and if D satisfies the GW relation, we have

$$\text{index}(D) = a^d \sum_x q(x), \qquad q(x) \equiv -\tfrac{1}{2} a \operatorname{tr}\{\gamma_5 D(x,x)\}$$

in all dimensions

The field $q(x)$ is

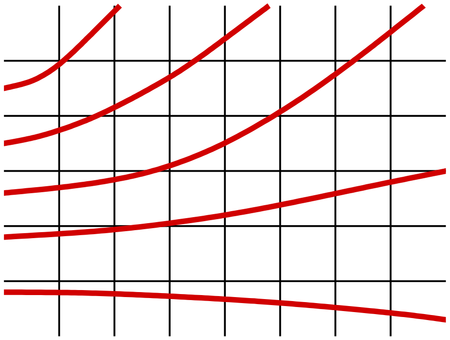
- $\star$ local & gauge-invariant (if $\|1 - U_{\Box}\| < \epsilon$)
- $\star$ pseudo-scalar under $\mathrm{O}(d, \mathbb{Z})$
- $\star$ topological

Indeed

$$a^4 \sum_x \delta q(x) = \tfrac{1}{2}\mathrm{Tr}\{\delta\hat{\gamma}_5\}, \qquad \hat{\gamma}_5 = \gamma_5(1 - aD)$$

$$(\hat{\gamma}_5)^2 = 1 \quad \Rightarrow \quad \{\hat{\gamma}_5, \delta\hat{\gamma}_5\} = 0 \quad \Rightarrow \quad \mathrm{Tr}\,\{\delta\hat{\gamma}_5\} = 0$$

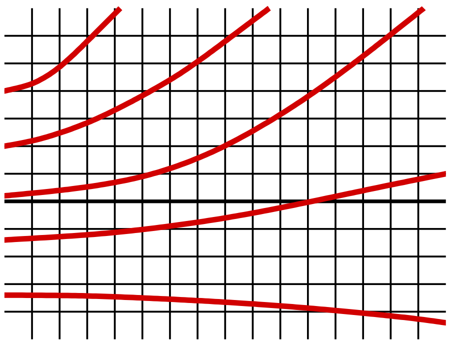
Classical continuum limit



- Consider a gauge field $A_\mu(x)$ on $\mathbb{R}^d$
- Superimpose a lattice with spacing a
- The lattice gauge field

$$U(x,\mu) = \mathrm{Pexp}\left\{\int_{x+a\hat{\mu}}^{x} \mathrm{d}z_\nu A_\nu(z)\right\}$$

then approximates the continuum field



In the limit $a \to 0$

$$q(x)\mathrm{d}x_0 \ldots \mathrm{d}x_{d-1} = -\mathrm{ch}_n(x) + \mathrm{O}(a)$$

Kikukawa & Yamada '98, Fujikawa '98, Suzuki '98, Adams '98,'01, ...

Fujiwara, Nagao & Suzuki '02

Chern polynomials

In $d = 8$ dimensions, for example,

$$\mathrm{ch}_2\,\mathrm{ch}_2 \quad \text{and} \quad \mathrm{ch}_4$$

represent different cohomology classes

All these can be obtained by considering product representations

$$R(X) = X \otimes 1 + 1 \otimes X$$

$$\mathrm{tr}\left\{R(X)^4\right\} = 6\,\mathrm{tr}X^2\,\mathrm{tr}X^2 + 2\,\mathrm{tr}1\,\mathrm{tr}X^4 \qquad \text{(assuming } \mathrm{tr}X = 0\text{)}$$

$$q(x)\mathrm{d}x_0 \ldots \mathrm{d}x_7 = -\mathrm{ch}_2(x)\mathrm{ch}_2(x) - 2\,\mathrm{tr}1\,\mathrm{ch}_4(x) + \mathrm{O}(a)$$

Summary

⋆ The NN theorem can effectively be bypassed

- Exact chiral symmetry
- Weyl fermions
- Index theorem

⋆ GW relation $\leftrightarrow$ topology of field space

⋆ Lattice fields representing the Chern polynomials