Chiral symmetry, the topological charge and the Yang–Mills gradient flow

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Based in part on work done with Peter Weisz

BNL & RBRC Joint Workshop "DWQ@25", 13-17 December 2021

Topological charge beyond the semi-classical level

Conceptually challenging, since

- Fields are typically nowhere continuous
- $\langle q(x)q(0)\rangle \underset{x\to 0}{\sim} |x|^{-8} \quad \Rightarrow \quad \langle Q^n\rangle$ not obviously well defined
- Field space in LQCD is connected

Often considered to be a "lattice problem" ...

Density-chain formulae

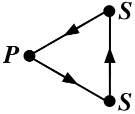
Giusti, Rossi & Testa '04
ML '04

Ginsparg & Wilson '82
Kaplan '92, . . .

Consider 4d LQCD with exact chiral symmetry

Topological charge $=$ chiral anomaly

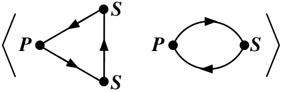
$$Q = \frac{a}{2}\mathrm{Tr}\{\gamma_5 D\} = m_1 m_2 m_3 \sum_{x,y,z} \underbrace{\left[P_{12}(x) S_{23}(y) S_{31}(z)\right]_{\mathrm{Wick}}}$$



For example

Assuming 5 or more quark flavours

$$\langle Q^2 \rangle = m_1 \dots m_5 \times$$



- Short-distance singularities are integrable
- Does not require renormalization

Well-defined <u>universal</u> formula for the topological susceptibility!

Flowed topological charge

ML '10

Weisz & ML '11

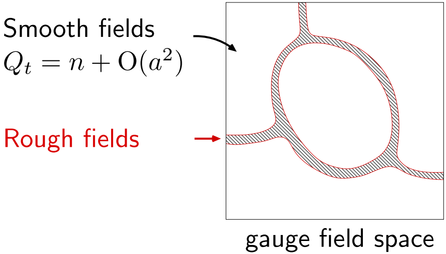
Hieda, Makino & Suzuki '17

At gradient-flow time $t > 0$

- No short-distance singularities
- The topological charge Q_t does not require renormalization
- In the continuum limit, $\partial_t \langle Q_t \ldots \rangle = 0$

In particular, the moments $\langle Q_t^n \rangle$ are well defined and independent of t

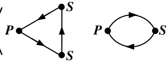
Moreover, as $a \to 0$ the gauge field becomes "smooth" with probability $1 - \mathrm{O}(a^2)$



$\Rightarrow$ Topological sectors emerge dynamically!

Gradient flow vs density chains

Does the equality

$$\langle Q_t^2 \rangle = m_1 \dots m_5 \times \Big\langle$$


$$\Big\rangle$$

hold beyond the semi-classical level?

Topology of field space $\leftrightarrow$ chiral anomaly

Cè, Consonni, Engel & Giusti '15 (YM theory)
ML '21 (full QCD)

Small flow-time expansion

As $t \to 0$

$$q_t(x) \sim c_1(t)\phi_1(x) + c_2(t)\phi_2(x) + \mathrm{O}(t)$$

where

$$\phi_1(x) = q(x) + X_A \partial_\mu A_\mu(x)$$

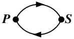
$$\phi_2(x) = Z_A \partial_\mu A_\mu(x)$$

Moreover, using perturbation theory and the RG

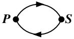
$$\lim_{t\to 0} c_1(t) = 1, \qquad \lim_{t\to 0} c_2(t) = 0$$

Hieda & Suzuki '16; ML & Weisz '21

As a consequence

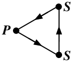
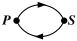
$$\langle Q_t^2 \rangle = \langle Q_t Q_s \rangle = \langle Q_t Q \rangle = m_4 m_5 \times \Big\langle Q_t$$  $$\Big\rangle$$

The correlation function

$$\Big\langle Q_t$$  $$\Big\rangle = \int_{x,y,z} \langle q_t(x) P_{12}(y) S_{21}(z) \rangle$$

however develops additional short-distance singularities as $t \to 0$

Must control these to be able to establish the identity

$$\langle Q_t^2 \rangle = m_1 \ldots m_5 \times \Big\langle$$   $$\Big\rangle$$

Generating function for density chains

Consider a complex mass matrix M

$$S_F = \int_x \overline{\psi}(x)\{\not{D} + MP_- + M^\dagger P_+\}\psi(x), \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$\partial^S_{rs} S_F = \int_x S_{rs}(x), \qquad \partial^P_{rs} S_F = \int_x P_{rs}(x)$$

$\Rightarrow$ The free energy

$$F(M) = -\ln\{Z(M)\}$$

generates density chains, e.g.

$$\partial^P_{12}\partial^S_{23}\partial^S_{31} F(M) = \int_{x,y,z} \langle P_{12}(x) S_{23}(y) S_{31}(z)\rangle_{\rm c}$$

Exact flavour symmetries

For any anti-Hermitian flavour matrix λ

$$\delta_\lambda^V M = [\lambda, M] \quad \Rightarrow \quad \delta_\lambda^V F(M) = 0$$

$$\delta_\lambda^A M = \{\lambda, M\} \quad \Rightarrow \quad \delta_\lambda^A F(M) = -2\,\mathrm{tr}\{\lambda\}\langle Q\rangle$$

Renormalization $(N_\mathrm{f} \geq 5)$

$F(M)$ requires additive renormalization

$$\Delta F(M) = V\left[\frac{z_0}{a^4} + \frac{z_1}{a^2}\mathrm{tr}\{M^\dagger M\} + z_2\,\mathrm{tr}\{M^\dagger M\}^2 + z_3\,\mathrm{tr}\{(M^\dagger M)^2\}\right]$$

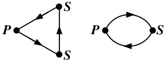
$\Rightarrow$ *$F + \Delta F$ has the same symmetry properties as F*

Final steps

Free energy $\leftrightarrow$ density-chain formula at $a > 0$

$$\Big[\delta_\lambda^A \delta_\eta^A \{F(M) + \Delta F(M)\}\Big]_{\text{diagonal } M} = 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q^2\rangle$$

$$= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\, m_1 \ldots m_5 \times \Big\langle$$



$$\Big\rangle$$

Final steps

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$$\left[\delta^A_\lambda \delta^A_\eta \{F(M) + \Delta F(M)\}\right]_{\text{diagonal } M} \cancel{= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q^2 \rangle}$$

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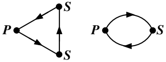

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Final steps

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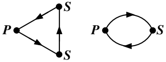
$$=4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\,m_1\ldots m_5\times\Big\langle$$  $$\Big\rangle$$

Free energy $\leftrightarrow$ gradient-flow formula at $a=0$ and any M

$$(1)\quad 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_t^2\rangle_\mathrm{c}=4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_tQ\rangle_\mathrm{c}=-2\,\mathrm{tr}\{\eta\}\delta_\lambda^A\langle Q_t\rangle$$

$$(2)\quad -2\,\mathrm{tr}\{\eta\}\langle Q_t\rangle=-2\,\mathrm{tr}\{\eta\}\langle Q\rangle=\delta_\eta^A\{F(M)+\Delta F(M)\}$$

$$\Rightarrow\quad 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_t^2\rangle_\mathrm{c}=\delta_\lambda^A\delta_\eta^A\{F(M)+\Delta F(M)\}$$

Conclusions

The equality of the density-chain and gradient-flow formulae

- ⋆ *Relates the chiral anomaly to the topology of field space at the fully non-perturbative level*
- ⋆ *Provides the definitive justification for using the flow-formula for the topological susceptibility*

Would be difficult to show w/o formulations of LQCD preserving chiral symmetry!