Demystifying lattice QCD computations of α_s

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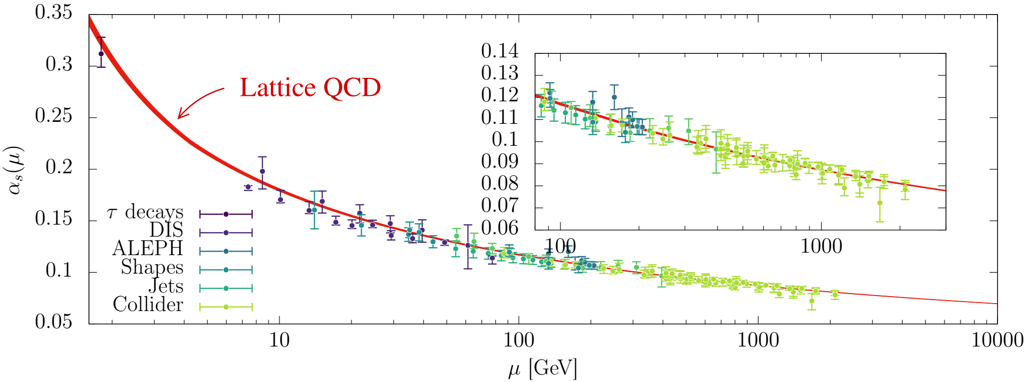
Dalla Brida et al. (ALPHA Collaboration), Nature **652** (2026) 328 [arXiv:2501.06633]



$$\underbrace{\tfrac{1}{3}(2f_K + f_\pi),\ M_\pi,\ M_K,\ M_D,\ M_B}_{\text{physics input}} \longrightarrow \alpha_s(M_Z) = 0.11876(58)$$

Outline

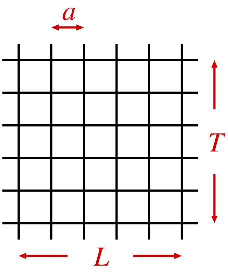
- $*$ Perturbative QCD $\subset$ Lattice QCD
- $*$ Asymptotic freedom & the continuum limit
- $*$ If we had infinite computing power . . .
- $*$ A strange world: the femto-universe
- $*$ Stepping up the energy scale
- $*$ Heavy quark decoupling revisited

Lattice QCD

= QCD at imaginary time with a UV regularisation

Lattice action = a discretised version of

$$S = \int \mathrm{d}^4x \left\{ \frac{1}{4g_0^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{q=u,d,s,\ldots} \bar{q} \left(\not{D} + m_{q,0} \right) q \right\}$$



Parameters

- $\boldsymbol{g_0}$: bare (i.e. unrenormalised) coupling
- $\boldsymbol{m_{q,0}}$: bare quark masses
- $\boldsymbol{a}$: lattice spacing
- $\boldsymbol{L}$: spatial size of the lattice
- $\boldsymbol{T}$: time extent of the lattice

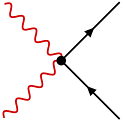
Perturbative QCD $\subset$ Lattice QCD

Lattice perturbation theory

- Momenta are restricted to the Brillouin zone

$$|p_\mu| \leq \pi/a$$

- Propagators & vertices are not rational functions
- Vertices with any number of gluon legs
- Gauge fixing as usual, exact BRS symmetry



$$= \mathrm{O}(a)$$

PhD of Thomas Reisz (1987/88):

Power-counting theorem for lattice Feynman diagrams

Proof of the renormalisability of lattice QCD

Renormalisation proceeds as usual

$$g^2 = g_0^2 Z, \qquad Z = 1 + \{z_{1,0} + z_{1,1}\ln(a\mu)\}g_0^2 + \dots$$

Minimal subtraction: $z_{1,0} = z_{2,0} = \dots = 0$

⇒ *Continuum limit exists and coincides with standard perturbative QCD up to a finite renormalisation of the parameters and fields*

In particular: $\alpha \equiv \dfrac{g^2}{4\pi} = \alpha_s + \underset{\substack{\uparrow\\ \text{computable}}}{c_1}\alpha_s^2 + \dots$

Asymptotic freedom & the continuum limit

At fixed g and μ,

$$g_0^2 = g^2 - \{z_{1,0} + z_{1,1}\ln(a\mu)\}g^4 + \dots$$

depends on the lattice spacing

The RGE then leads to the exact asymptotic expression

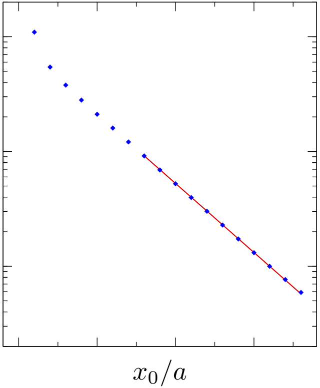
$$g_0^2 \underset{a\mu\to 0}{=} -\frac{1}{2b_0\ln(a\Lambda)} + \dots$$

$$\Lambda = \mu\phi(g) = \underset{\substack{\uparrow \\ \text{exactly calculable}}}{k}\Lambda_{\overline{\mathrm{MS}}}$$

⇒ *Provides a link between the lattice theory and collider physics*

If we had infinite computing power . . .

How meson masses are computed in LQCD



- Choose g_0, the quark masses $am_{q,0}$ and a large lattice
- Introduce interpolating fields for the mesons, e.g.

$$\pi^+(x) = \bar{d}(x)\gamma_5 u(x), \qquad \pi^-(x) = \bar{u}(x)\gamma_5 d(x)$$

- Compute

$$a^3 \sum_{\boldsymbol{x}} \langle \pi^+(x)\pi^-(0) \rangle \underset{x_0 \to \infty}{\propto} \mathrm{e}^{-M_\pi x_0}$$

using numerical simulations

⇒ *$aM_\pi, aM_K, \ldots$ at the chosen values of g_0 and $am_{q,0}$*

(af_π and af_K are obtained similarly)

What is the lattice spacing in fm?

Fix g_0 and tune the quark masses $am_{0,u}, am_{0,d}, \ldots$ such that

$$M_\pi/f_\pi, M_K/f_\pi, \ldots$$

assume their physical values. Then

$$a = \frac{af_\pi}{f_\pi} = af_\pi \times 1.513(2)\,\mathrm{fm}$$

Computation of α_s

Recall

$$g_0^2 \underset{a\mu \to 0}{=} -\frac{1}{2b_0 \ln(a\Lambda)} + \ldots \quad \text{and thus } a \ll 1\,\mathrm{fm} \text{ if } g_0 \ll 1$$

⇒ May extract Λ (in MeV) at these couplings

$$\Rightarrow \Lambda_{\overline{\mathrm{MS}}} \Rightarrow \alpha_s(M_Z) = \frac{1}{8\pi b_0 \ln(M_Z/\Lambda_{\overline{\mathrm{MS}}})} + \ldots$$

The “window problem”

Should have

$$\left.\begin{array}{l}\blacktriangleright\ L \geq 6\,\mathrm{fm}\ \textit{to suppress finite-volume effects}\\[2ex] \blacktriangleright\ 1/a = \mathrm{O}(100)\,\mathrm{GeV}\ \textit{for } g_0 \textit{ to be sufficiently small}\end{array}\right\}\quad L/a = \mathrm{O}(3000)$$

In practice

$L/a \leq 128\ldots256 \Rightarrow 1/a \leq 4.2\ldots8.4\,\mathrm{GeV}$ @ $L = 6\,\mathrm{fm}$

Computational effort $\propto 1/a^{6\ldots7}$

A strange world: the femto-universe

Consider QCD in an L^4 box with $L \ll 1\,\mathrm{fm}$

- Perturbative regime at scale $\mu = 1/L$
- No hadrons — a world of quarks and gluons

James D. Bjorken

QCD in the box renormalises in the same way as in infinite volume with L-independent renormalisation constants

Holds for various boundary conditions

- Periodic in space & time
- Periodic in space, Dirichlet in time

QCD on femto-lattices

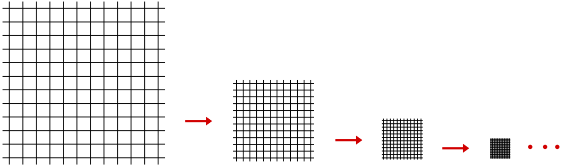
Can be simulated straightforwardly

$\Rightarrow$ At sufficiently small L and $a \ll L$ given in fm

$$g_0^2 = -\frac{1}{2b_0 \ln(a\Lambda)} + \ldots \;\rightarrow\; \Lambda_{\overline{\mathrm{MS}}}\,[\mathrm{MeV}] \;\rightarrow\; \alpha_s(M_Z)$$

How to connect QCD at large L to the femto-universe

- Start from a lattice with large L
- Set $m_q = 0$ at fixed g_0 (i.e. fixed a)
- Use "step scaling" to go from $L, a \rightarrow L/2, a/2 \rightarrow \ldots$



Step scaling

Lüscher, Weisz & Wolff, Nucl. Phys. B359 (1991) 221

Each cycle proceeds in two steps

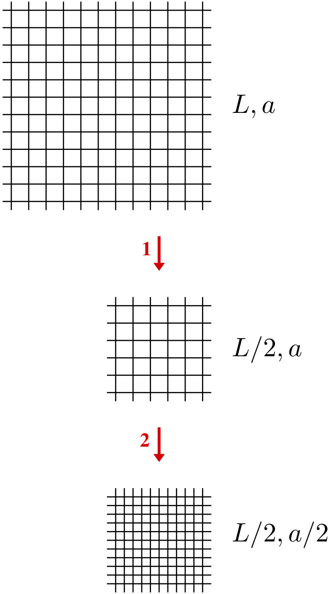
1. Size reduction

Reduce $L \to L/2$ at fixed g_0 and thus fixed a [fm]

2. Renormalisation step

Reduce $a \to a/2$ and adjust g_0 such that
"the physics is unchanged"

⇒ box size $L/2$ [fm] remains the same



Couplings used in the renormalisation step

$\boldsymbol{\alpha}_{\mathbf{SF}}$: Schrödinger-functional coupling

Lüscher, Narayanan, Weisz & Wolff (1992)

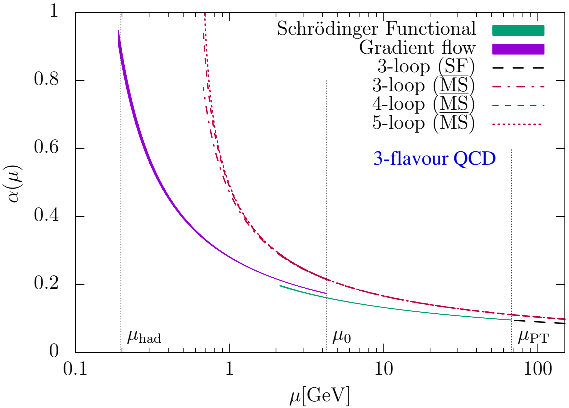
$\boldsymbol{\alpha}_{\mathbf{GF}}$: based on the YM gradient flow

Fritzsch & Ramos (2013)

Properties

- Renormalisation scale $\mu = 1/L$
- Non-perturbatively defined
- $\alpha_{\overline{\rm MS}} = \alpha + k_1\alpha^2 + \ldots$

Bruno et al. (ALPHA Collab.), PRL 119 (2017) 102001



$$\Rightarrow \Lambda^{(3)}_{\overline{\rm MS}} = 341(12)\,{\rm MeV}$$

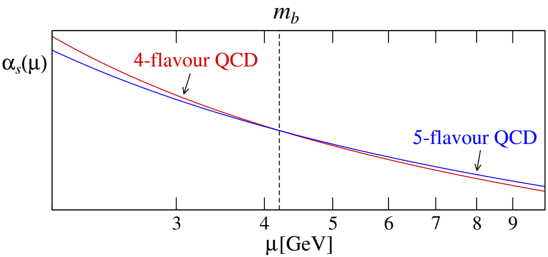
How can the heavy quarks be included?

Another window problem . . .

$$m_c = 1.27\,\mathrm{GeV}, \quad M_D = 1.87\,\mathrm{GeV}, \quad c\bar{c}\ \text{threshold} \sim 3\,\mathrm{GeV}$$

$$m_b = 4.18\,\mathrm{GeV}, \quad M_B = 5.28\,\mathrm{GeV}, \quad b\bar{b}\ \text{threshold} \sim 9\,\mathrm{GeV}$$

α_s depends on the number of quark flavours



$$\rightarrow \quad \alpha_s^{(4)}(M_Z) < \alpha_s^{(5)}(M_Z)$$

Heavy quark decoupling

Appelquist & Carazzone (1975)

At energies $E \ll m_b$

$$\mathrm{QCD}^{(5)} = \underset{\substack{\uparrow \\ \text{with adjusted parameters}}}{\mathrm{QCD}^{(4)}} + \mathrm{O}(E^2/m_b^2)$$

In particular,

$$g^2\big|_{\text{4-flavour QCD}} = g^2 + \{a_1 + b_1 \ln(m_b/\mu)\}g^4 + \ldots$$

↓

is known up to 4 loops

Bernreuther & Wetzel (1982)

⋮

Chetyrkin, Kühn & Sturm (2005)

Schröder & Steinhauser (2005)

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(4)}}{\Lambda_{\overline{\mathrm{MS}}}^{(5)}} = 1.4255 - \overset{\text{2-loop}}{0.0308} - \overset{\text{3-loop}}{0.0029} - \overset{\text{4-loop}}{0.0005}$$

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\Lambda_{\overline{\mathrm{MS}}}^{(4)}} = 1.1875 - 0.0323 - 0.0060 - 0.0024$$

Non-perturbative and E^2/m_c^2 effects on $\Lambda^{(4)}_{\overline{\mathrm{MS}}}$

. . . were shown to be on the level of a fraction of the error of $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$

Bruno et al. (ALPHA Collab.), Phys. Rev. Lett. 114 (2015) 102001

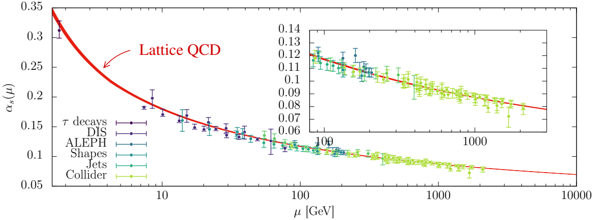
Athenodorou et al. (ALPHA Collab.), Nucl. Phys. B943 (2019) 114612

→ talk by Alberto Ramos later today

Concluding remarks

$$\underbrace{\tfrac{1}{3}(2f_K+f_\pi),\ M_\pi,\ M_K,\ M_D,\ M_B}_{\displaystyle\Big\downarrow}$$

$$\alpha_s^{(5)}(M_Z) = 0.11876(58)$$



- ★ *Computation links low-energy hadron to collider physics*
- ★ *A beautiful piece of evidence that QCD is the theory of the strong interactions at all energies*

Backup slides

Renormalisation in finite volume

The singularities of propagators on an L^4 box with periodic boundary conditions do not depend on L

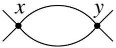
In infinite volume

$$G(x,y;\infty) \underset{x\to y}{\propto} \frac{1}{(x-y)^2} + \ldots$$

but

$$\left(-\Box + m^2\right)\{G(x,y;L) - G(x,y;\infty)\} = 0$$ ⇒ difference is regular at $x = y$

Example: massive ϕ^4 theory



$$= G(x,y;L)^2 = G(x,y;\infty)^2 + \text{integrable}$$

Kopper, Müller & Reisz, Annals Henri Poincaré 2 (2001) 387 [ϕ^4 to all orders]

Bode, Weisz & Wolff, Nucl. Phys. B576 (2000) 517 [QCD with SF bc up to 2 loops]

Definition of α_{SF}

Lüscher, Narayanan, Weisz & Wolff, Nucl. Phys. B384 (1992) 168

Consider an L^4 cylindrical space-time

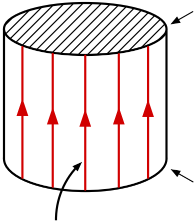
Free energy of the induced field

$$-\ln \mathcal{Z} = \underbrace{\frac{1}{g_0^2}\Gamma_0[E]}_{\text{classical action}} + \Gamma_1[E] + g_0^2\Gamma_2[E] + \ldots$$

SF coupling at renormalisation scale $\mu = 1/L$

$$g_{\mathrm{SF}}^2 = -\frac{\partial\Gamma_0}{\partial E}\left\{\frac{\partial \ln \mathcal{Z}}{\partial E}\right\}^{-1}$$

$$= g_{\overline{\mathrm{MS}}}^2 + \ldots$$



$x_0 = L,\ A_k = C'_k$

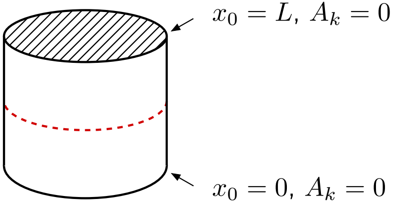
$x_0 = 0,\ A_k = C_k$

induced chromo-electric field $E = (C' - C)/L$

Definition of α_{GF}

Fritzsch & Ramos, JHEP 10 (2013) 008

Consider an L^4 cylindrical space-time



Yang–Mills gradient flow

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \qquad B_\mu|_{t=0} = A_\mu$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

GF coupling at renormalisation scale $\mu = 1/L$

$$\alpha_{\mathrm{GF}} = \text{constant} \times \left\{ t^2 \left\langle G^a_{kl} G^a_{kl} \right\rangle \right\}_{x_0 = L/2, \sqrt{8t} = 0.3 \times L}$$

$$= \alpha_{\overline{\mathrm{MS}}} + \ldots$$