Introduction to multilevel algorithms

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School on "Computing Challenges in HEP @Exascale"
Benasque, September 19–23, 2022

Commented List of References

1. Basic multilevel strategy

In lattice gauge theory, an important error reduction through a multilevel algorithm was first achieved by

[1] G. Parisi, R. Petronzio, F. Rapuano, *A measurement of the string tension near the continuum limit*, Phys. Lett. B 128 (1983) 418

in a computation of Wilson loops in the SU(3) gauge theory. The algorithm became known as the "multihit method" for Wilson lines and was widely adopted.

Nearly 20 years later, a more efficient multilevel algorithm for large Wilson loops was described in

[2] M. Lüscher, P. Weisz, *Locality and exponential error reduction in numerical lattice gauge theory*, JHEP 09 (2001) 010.

The 2-level algorithm for the spin-field theory discussed in the first lecture coincides with the one proposed in

[3] H. B. Meyer, *Locality and statistical error reduction on correlation functions*, JHEP 01 (2003) 048

for correlation functions of small Wilson loops.

Multilevel algorithms are normally not used in spin-field theories, since the exponential loss of precision can be avoided in this case through cluster and other collective algorithms. An algorithm of this kind first appeared in

[4] U. Wolff, *Collective Monte Carlo updating for spin systems*, Phys. Rev. Lett. 62 (1989) 361

and an associated improved observable for the spin 2-point function was introduced in

[5] U. Wolff, *Asymptotic freedom and mass generation in the O(3) non-linear σ-model*, Nucl. Phys. B 334 (1990) 581.

To date such collective algorithms are however not available for non-Abelian gauge theories in four dimensions.

2. Exponential loss of significance in QCD

The relation between the statistical errors of hadron correlation function and the hadron spectrum was first discussed in

[6] G. Parisi, *The strategy for computing the hadronic mass spectrum*, Phys. Rept. 103 (1984) 203

and later in more detail in

[7] G. P. Lepage, *The analysis of algorithms for lattice field theory*, in: *From actions to answers*, Eds. T. DeGrand, D. Toussaint (World Scientific, Singapore, 1990).

While the argumentation in these papers is not completely rigorous, there is ample numerical evidence for the correctness of the derived results.

3. Multilevel strategy in QCD

The multilevel algorithm discussed in the lectures was put forward in

[8] M. Cè, L. Giusti, S. Schaefer, *Domain decomposition, multi-level integration and exponential noise reduction in lattice QCD*, Phys. Rev. D 93 (2016) 094507,

[9] M. Cè, L. Giusti, S. Schaefer, *A local factorization of the fermion determinant in lattice QCD*, Phys. Rev. D 95 (2017) 034503.

Since this seminal work appeared, various numerical studies were performed and the algorithm has seen further developments. The most recent published articles are

[10] M. Dalla Brida, L. Giusti, T. Harris, M. Pepe, *Multi-level Monte Carlo computation of the hadronic vacuum polarization contribution to* $(g_\mu - 2)$, Phys. Lett. B 816 (2021) 136191,

[11] L. Giusti, M. Saccardi, *Four-dimensional factorization of the fermion determinant in lattice QCD*, Phys. Lett. B 829 (2022) 137103.

Many related conference reports are quoted in these two papers.

4. Multiboson representation

The multiboson representation was introduced in

[12] M. Lüscher, *A new approach to the problem of dynamical quarks in numerical simulations of lattice QCD*, Nucl. Phys. B 418 (1994) 637.

In the lectures, a complex variant of the multiboson representation is discussed following

[13] A. Boriçi, P. de Forcrand, *Systematic errors of Lüscher's fermion method and its extensions*, Nucl. Phys. B 454 (1995) 645,

[14] A. Borrelli, P. de Forcrand, A. Galli, *Non-Hermitian exact local bosonic algorithm for dynamical quarks*, Nucl. Phys. B 477 (1996) 809.

The state of the multiboson representation and its use for numerical simulations of QCD at the end of the 1990's was summarized in

[15] P. de Forcrand, *The multiboson method*, Parallel Comput. 25 (1999) 1341.

In the following years, the multiboson representation did not receive much attention until it was used in ref. [9] to represent the determinant of the SAP-preconditioned Dirac operator.

5. Schwarz alternating procedure (SAP)

In its original form, the SAP is an iterative algorithm for the solution of elliptic linear differential equations. Nowadays it is mostly used as preconditioner for difference operators in two and three dimensions such as the discretized Laplace operator. See

[16] Y. Saad, *Iterative methods for sparse linear systems*, 2nd ed. (SIAM, Philadelphia, 2003),

for example, for an introduction to the subject (an older version of the book is freely available at `https://www-users.cs.umn.edu/~saad/`).

In lattice QCD, the application of the non-overlapping SAP was first considered in

[17] M. Lüscher, *Solution of the Dirac equation in lattice QCD using a domain decomposition method*, Comput. Phys. Commun. 156 (2004) 209.

Preconditioners of the Dirac operator can lead to factorizations of the quark determinant, which permit the short-distance effects of the quarks to be separated from their long-distance effects. The simulation algorithm described in

[18] M. Lüscher, *Schwarz-preconditioned HMC algorithm for two-flavour lattice QCD*, Comput. Phys. Commun. 165 (2005) 199

implements this general strategy using the non-overlapping SAP as preconditioner.

The factorization of the quark determinant discussed in the lectures coincides with the one discussed in ref. [9], but the connection with the overlapping SAP was discovered only after this paper appeared and is therefore not mentioned there.

6. Factorization of correlation functions

Multilevel simulations of QCD require the factorization of both the quark determinant and the observables. In the case of the meson and baryon correlation functions, the associated quark-line diagrams can be approximately factorized in various ways, the approximation error being corrected at level 0 of the simulations.

Such factorizations were discussed already in the first papers on the subject [8,9] and later in many publications, including

[19] L. Giusti, M. Cè, S. Schaefer, *Multi-boson block factorization of fermions*, EPJ Web Conf. 175 (2018) 01003,

[20] L. Giusti, T. Harris, A. Nada, S. Schaefer, *Multi-level integration for meson propagators*, PoS LATTICE2018 (2018) 028

and ref. [10].

At present the factorization of the quark-line diagrams contributing to meson and baryon correlation functions is still a research topic with ample room for further consolidation and development. The nearly exact factorization of the vector meson propagator discussed in the lectures, for example, has not yet been studied numerically and its viability in practice thus remains to be assessed.