Introduction to multilevel algorithms

II: Multilevel strategy in QCD

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As a representative case, consider . . .

Lattice QCD in 4d with 2 flavours of light mass-degenerate Wilson quarks

$$S(U,\psi,\overline{\psi}) = S_{\rm G}(U) + S_{\rm F}(U,\psi,\overline{\psi})$$

$S_{\rm G}(U) =$ Wilson plaquette action

$$S_{\rm F}(U,\psi,\overline{\psi}) = \sum_x \overline{\psi}(x)(D\psi)(x), \qquad D = \text{Wilson–Dirac operator}$$

The multilevel strategy to be discussed however straightforwardly extends to the theory with improved actions and more quarks

Meson correlation functions

After integrating over the quark fields

$$\langle\overline{\psi}(x)\Gamma_1\psi(x)\,\overline{\psi}(y)\Gamma_2\psi(y)\rangle$$

$$= -\frac{1}{\mathcal{Z}}\int \mathrm{D}[U]\,\underbrace{\mathrm{e}^{-S_{\mathrm{G}}(U)}|\det(D)|^2}_{\text{distribution } P(U)} \times \underbrace{\mathrm{tr}\{\Gamma_1 S(x,y)\Gamma_2 S(y,x)\}}_{\text{observable } \mathcal{O}(U)}$$

⇒ *The multilevel strategy appears to be inapplicable, since $P(U)$ and $\mathcal{O}(U)$ are non-local functions of U*

⇒ *First need to find a way to factorize these expressions*

Cè, Giusti & Schaefer '16 (observable), '17 (distribution)

Exponential decay of the quark propagator $S(x,y)$

In the case of the pion propagator

$$\mathcal{O}(U) = \mathrm{tr}\{S(x,y)^\dagger S(x,y)\} \geq 0$$

At large distances

$$\langle \mathcal{O} \rangle \propto \mathrm{e}^{-M_\pi |x-y|}, \qquad \sigma(\mathcal{O}) \propto \mathrm{e}^{-M_\pi |x-y|}$$

$$\Rightarrow \quad S(x,y) \propto \mathrm{e}^{-\frac{1}{2} M_\pi |x-y|} \pm \text{statistical fluctuations}$$

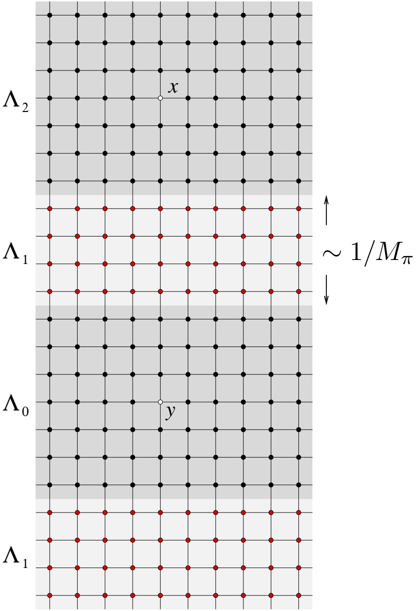
Multilevel strategy

To decouple Λ_0 from Λ_2 use

- ⋆ Overlapping SAP preconditioner

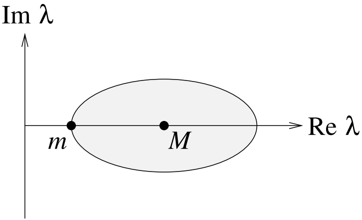
$$\det D \to \frac{\det DM_{\mathrm{sap}}}{\det M_{\mathrm{sap}}}$$

- ⋆ Multiboson representation for the residual term $\det DM_{\mathrm{sap}}$



Multiboson representation

Spectrum of D



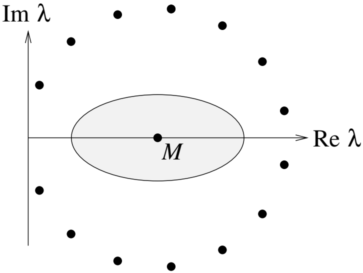
Polynomial approximation

$$\frac{M}{D} = \frac{1}{1-(1-D/M)} \simeq \sum_{k=0}^{N} (1-D/M)^k \equiv P_N(D)$$

Factorization of $P_N(D)$ for even N

$$P_N(D) \propto \prod_{k=1}^{N/2} (D - Mz_k)(D - Mz_k^*)$$

$$z_k = 1 - \exp\left\{2\pi i \frac{k}{N+1}\right\}$$



With $N/2$ pseudo-fermion fields $\phi_k(x)$

$$\det(D/M) \simeq \frac{1}{\det P_N(D)} \propto \int \mathrm{D}[\phi]\, \mathrm{e}^{-S_{\mathrm{pf}}(U,\phi)}$$

$$S_{\mathrm{pf}}(U,\phi) = \sum_{k=1}^{N/2} \|(D - Mz_k)\phi_k\|^2 \qquad = \text{ local!}$$

Systematic error & reweighting

Error correction through reweighting

$$\langle \mathcal{O} \rangle = \langle \mathcal{O}\hat{W} \rangle_N, \qquad \hat{W} = \frac{W}{\langle W \rangle_N}, \qquad W = \det\{DP_N(D)\}^2$$

Upper bound on the reweighting effect

$$\langle \mathcal{O} \rangle - \langle \mathcal{O} \rangle_N = \langle (\mathcal{O} - \langle \mathcal{O} \rangle_N)(\hat{W} - \langle \hat{W} \rangle_N) \rangle_N$$

$$|\langle \mathcal{O} \rangle - \langle \mathcal{O} \rangle_N| \leq \sigma(\mathcal{O})\sigma(\hat{W})$$

$\Rightarrow$ Effect is negligible if, say,

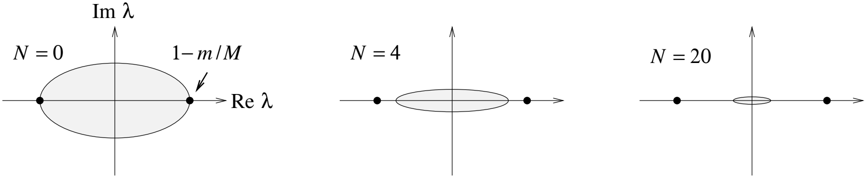
$$\sigma(\hat{W}) \leq 0.1 \times \sqrt{\frac{2\tau(\mathcal{O})}{N_{\mathrm{ms}}}}$$

Rate of convergence

Summing the geometric series ...

$$DP_N(D) = M\{1 - (1 - D/M)^{N+1}\}$$

Spectrum of $(1 - D/M)^{N+1}$



$$\det(DP_N(D)) \propto \exp\left\{\mathrm{Tr}\ln[1 - (1 - D/M)^{N+1}]\right\}$$

$$\hat{W} = 1 + \mathrm{O}\left(V \exp\{-(m/M)N\}\right)$$

That N must be increased like $1/m$ and slowly with V is empirically supported

By the end of the 1990's, the multiboson idea was abandoned

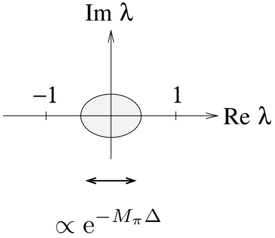
- $N \sim 100 \ldots 1000+$ in the cases of interest
- $\tau(\mathcal{O}) \propto N$
- Strong competition (HMC, ...)!

Multibosons & multilevel

In the multilevel algorithm

$$D \to DM_{\mathrm{sap}} = 1 - w$$

Spectrum of w depends on the width Δ of the inactive region Λ_1



⇒ *Need only $N \sim 10$ bosons*

⇒ *Problems are gone – may use the multiboson representation*