SF with GW and normal bc

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The normal bc on the quark fields are

$$P_+\psi(x) = \overline{\psi}(x)P_- = 0 \quad \text{at } x_0 = 0,$$

$$P_-\psi(x) = \overline{\psi}(x)P_+ = 0 \quad \text{at } x_0 = T, \qquad P_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_0)$$

Preserves the flavour symmetry, parity, charge conjugation, etc.

Do we need this?

- Non-perturbative renormalization
- Mixed action theory   Bär, Rupak & Shoresh '03

Not sure how to proceed, since

- The Dirac operator is not ultra-local
- Boundary conditions break chiral symmetry

Orbifold construction   Taniguchi '05f, Sint '05f

Works well with tm bc, but not too well with normal bc

The construction proposed here largely relies on

- ⋆ Universality arguments

  Boundary conditions arise dynamically

- ⋆ Symmetry considerations

  GW relation must be modified close to the boundaries

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In the massless continuum theory, the quark propagator satisfies

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 =$$

$$\int_{z_0=0} \mathrm{d}^3\boldsymbol{z}\, S(x,z)\gamma_5 S(z,y) + \int_{z_0=T} \mathrm{d}^3\boldsymbol{z}\, S(x,z)\gamma_5 S(z,y)$$

Suggests a modified GW relation

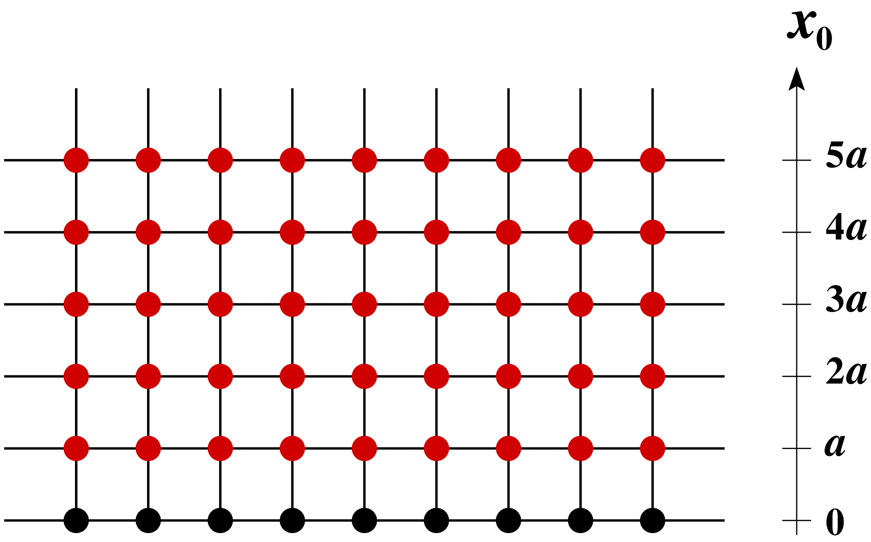
$$\{\gamma_5, D\} = aD\gamma_5 D + \gamma_5 P$$

$\uparrow$ supported on the boundaries

- *Chiral symmetry is preserved away from the boundaries*
- *Not obvious how to construct such Dirac operators*

Field theories on lattices with boundaries

Consider a free scalar field on the halfspace $x_0 \geq 0$



$$S = a^4 \sum_{x_0 \geq a} \sum_{\boldsymbol{x}} \tfrac{1}{2} \left\{ \partial_\mu \phi(x) \partial_\mu \phi(x) + m^2 \phi(x)^2 + c a^{-2} \delta_{x_0 a} \phi(x)^2 \right\}$$

$\Rightarrow$ in the continuum limit, the propagator $\langle \phi(x) \phi(y) \rangle$ satisfies

- Neumann bc if $c = 0$
- Dirichlet bc for any fixed $c > 0$

- ⋆ *Boundary conditions arise dynamically in the continuum limit*

- ⋆ *There are different universality classes of lattice theories, corresponding to different bc*

- ⋆ *Some bc are unstable against perturbations of the lattice action*

Natural boundary conditions

Close to the continuum limit, the bc are expected to be of the form

$$\mathcal{O}(x)\big|_{x_0=0} = 0, \qquad \mathcal{O}(x) \propto \{k_1\phi(x) + k_2 a\partial_0\phi(x) + \ldots\}$$

⇒ *In general, Neumann bc require a fine-tuning of the lattice action such that* $k_1 = 0$

⇒ *The natural bc are Dirichlet in this case* Symanzik '81

SF bc arise naturally in QCD if the lattice theory is local and if the obvious symmetries are preserved

(true up to a sign ambiguity that amounts to interchanging P_+ with P_-)

The problem is thus reduced to finding a lattice Dirac operator that

- ⋆ *acts on quark fields at* $0 < x_0 < T$
- ⋆ *is local and respects the lattice symmetries*
- ⋆ *coincides with (say) the Neuberger–Dirac operator at physical distances from the boundaries*

Lattice Dirac operator

First consider the infinite lattice

Wilson–Dirac operator

$$D_{\mathrm{w}} = \tfrac{1}{2}\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^*\nabla_\mu\}$$

Neuberger–Dirac operator

$$D = \frac{1}{\bar{a}}\{1 - A\left(A^\dagger A\right)^{-1/2}\}$$

$$A = 1 + s - aD_{\mathrm{w}}, \qquad \bar{a} = \frac{a}{1+s}$$

In the presence of the boundaries

SF Wilson–Dirac operator

$$D_{\mathrm{w}}\psi(x), \quad x_0 = a, 2a, \ldots, T-a, \quad \psi(x)|_{x_0=0} = \psi(x)|_{x_0=T} = 0$$

SF Neuberger–Dirac operator

$$D = \frac{1}{\bar{a}}\left\{1 - \tfrac{1}{2}(U + U^{\sim})\right\},$$

$$U = A\left(A^{\dagger}A + caP\right)^{-1/2}, \qquad U^{\sim} = \gamma_5 U^{\dagger}\gamma_5, \qquad c \geq 1$$

$\uparrow$

supported at $x_0 = a$ and $x_0 = T - a$

Note: $A^{\dagger}A$ has eigenvalues $\sim \mathrm{e}^{-\kappa T/a}$ (“3d domain-wall fermions”)

If we set

$$P\psi(x) = \frac{1}{a}\left\{\delta_{x_0 a} P_- \psi(x)|_{x_0=a} + \delta_{x_0\, T-a} P_+ \psi(x)|_{x_0=T-a}\right\}$$

the locality of D is guaranteed, since

$$R\left(A^\dagger A + aP\right)R = R\left\{A^\dagger A\right\}_{\text{infinite lattice}} R$$

$$R\psi(x) = \begin{cases} \psi(x) & \text{if } 0 < x_0 < T \\ 0 & \text{otherwise} \end{cases}$$

⇒ Range of locality is not larger than on the infinite lattice

In particular, the P–term removes the 3d domain-wall fermions

Other properties of D

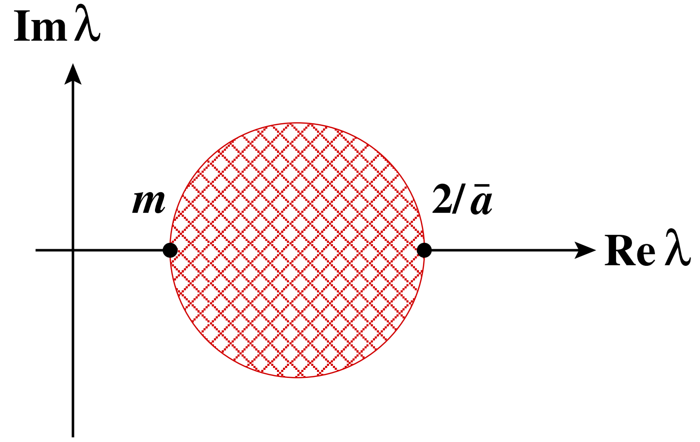
- *D respects all lattice symmetries and $\gamma_5 D$ is hermitian*

- *The determinant of the massive Dirac operator*

$$D_{\rm m} = (1 - \tfrac{1}{2}\bar{a}m)D + m$$

is positive if $0 < m \leq 2/\bar{a}$

- *The spectrum of $D_{\rm m}$ is contained in:*



- *At a distance d from the boundaries*

$$D(x,y) = D(x,y)_{\textit{Neuberger–Dirac}} + \mathrm{O}(\mathrm{e}^{-\kappa d/a})$$

where $\rho = a/\kappa$ is the localization range

- *In particular*

$$\gamma_5 D + D\gamma_5 = \bar{a} D\gamma_5 D + \Delta_B$$

Δ_B: local, supported in the vicinity of the boundaries

The SF with GW and normal bc is ready for use !

In lattice QCD, we have reached an interesting point . . .

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. . . we can see the mountain, but are not quite there yet!