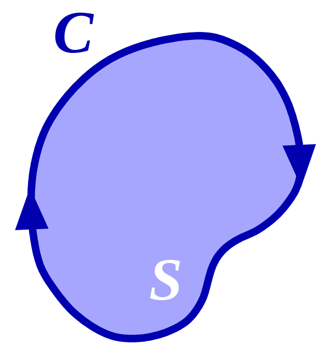
SU(N) gauge theories & the bosonic string

Martin Lüscher

CERN — Theory Division

Wilson loop $\simeq \int_{\text{surfaces}} \mathrm{e}^{-\sigma A(S)}$



- ⋆ Expected to hold when C is large
- ⋆ $\Rightarrow$ expansion in powers of $\sigma^{-1/2}$ about $S = S_{\text{min}}$

Nambu '79; M.L., Symanzik & Weisz '80

Niels Bohr Institute, Copenhagen, October 2004

Is this basically correct? If so . . .

- Exactly which string theory?
  - ◇ Alternative string actions (“rigid” string, etc.)
    Polyakov ’86, Savvidy & Savvidy ’93
  - ◇ String theories with fermionic modes
    Ramond ’71, Neveu & Schwartz ’71
- At which distances does string behaviour set in?

⇒ lattice gauge theory

Michael & Perantonis ’90; Juge, Kuti & Morningstar ’98ff

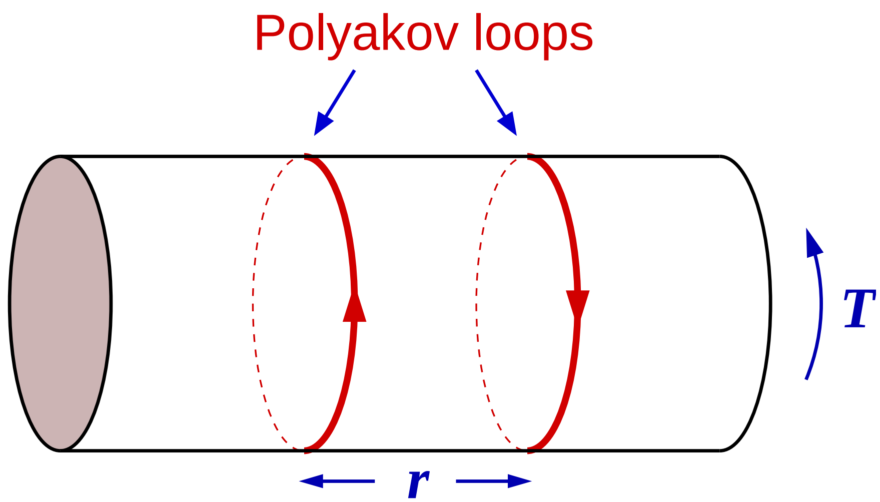
Caselle et al. ’97ff, Lucini & Teper ’01, Necco & Sommer ’02, M.L. & Weisz ’02

Caselle, Hasenbusch & Panero ’02ff

Polyakov loop correlation function

In gauge theories with any compact gauge group

$$\langle P(x)^* P(0)\rangle = \sum_{n=0}^{\infty} w_n \mathrm{e}^{-E_n T}, \qquad r \equiv |\vec{x}|$$



$E_0(r) \equiv V(r),\ w_0 = 1$

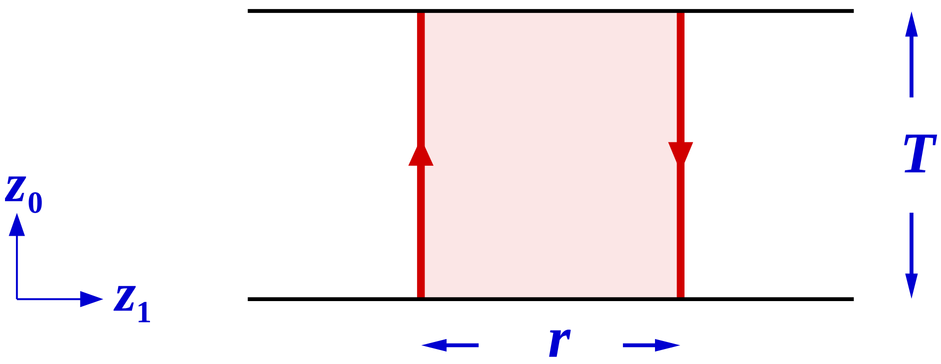
(static quark potential)

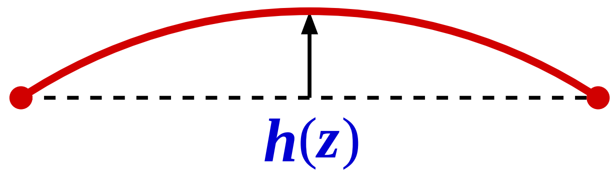
$E_n(r), n \geq 1,\ w_n \in \mathbb{Z}$

(excited states)

Parisi, Petronzio & Rapuano '83, Otha, Fugikawa & Ukawa '85, de Forcrand et al. '85, . . .

Effective string theory





$$\langle P(x)^* P(0) \rangle = \mathrm{e}^{-\sigma r T - \mu T} \times \int_{\text{fluctuations } h} \mathrm{e}^{-S_{\text{eff}}}$$

$$S_{\text{eff}} = \int_0^T \int_0^r \mathrm{d}z_0 \mathrm{d}z_1 \left\{ \frac{1}{2} (\partial h)^2 + \frac{1}{4} c\, (\partial h)^2 (\partial h)^2 + \ldots \right\}$$

To leading order

$$\langle P(x)^* P(0) \rangle = \mathrm{e}^{-\sigma r T - \mu T} \left[\det(-\Delta)\right]^{-\frac{1}{2}(d-2)}$$

$$\left[\det(-\Delta)\right]^{\frac{1}{2}} = \eta(q) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \qquad q \equiv \mathrm{e}^{-\pi T/r}$$

$\Rightarrow$ free-string energy spectrum

$$E_0 = \sigma r + \mu - \frac{\pi}{24 r}(d-2)$$

(d: space-time dimension)

$$E_n = E_0 + \frac{n\pi}{r}$$

$$w_0 = 1,\ w_1 = d - 2,\ \ldots$$

M.L., Symanzik & Weisz '80, M.L. '81, Ambjørn, Olesen & Peterson '84, . . .

Higher orders

- Interactions are non-renormalizable

- $\Rightarrow$ perturbation theory = expansion in powers of r^{-1}

$$S_1 = \tfrac{1}{4} b \int_0^T \mathrm{d}z_0 \left\{ (\partial_1 h \partial_1 h)_{z_1=0} + (\partial_1 h \partial_1 h)_{z_1=r} \right\}$$

$$E_0 = \sigma r + \mu - \frac{\pi}{24 r} (d-2) \left(1 + b/r + \ldots\right)$$

$$E_1 = E_0 + \frac{\pi}{r} \left(1 + b/r + \ldots\right)$$

- $\Rightarrow$ leading terms are universal

M.L. '81, M.L. & Weisz '02

Studying string behaviour in LGT

⋆ Ground state energy

Lucini & Teper '01

Necco & Sommer '02

M.L. & Weisz '02

$$V'(r) = \sigma + \mathrm{O}(r^{-2})$$

$$V''(r) = \frac{\pi}{12r^3}(d-2) + \mathrm{O}(r^{-4})$$

⋆ Low-lying excited states

Michael & Perantonis '90

Juge, Kuti & Morningstar '98ff

$$E_n = E_0 + \frac{n\pi}{r} + \mathrm{O}(r^{-2})$$

⋆ Wilson loop expectation values

Caselle et al. '97ff

Caselle, Hasenbusch & Panero '02ff

The principal difficulties are

- ! The signal

$$\langle PP \rangle \propto \mathrm{e}^{-\sigma r T}$$

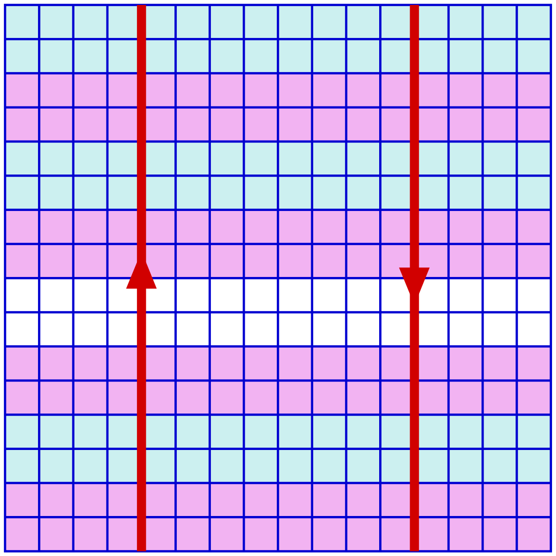
decreases exponentially ($\sim 10^{-25}$ at $a = 0.1\,\mathrm{fm}$, $rT = 5\,\mathrm{fm}^2$)

- ! The significance loss in

$$-\tfrac{1}{2} r^3 V''(r) = \frac{\pi}{24}\,(d-2) + \ldots$$

grows proportionally to $\sigma r^4 / a^2$

Multilevel algorithm



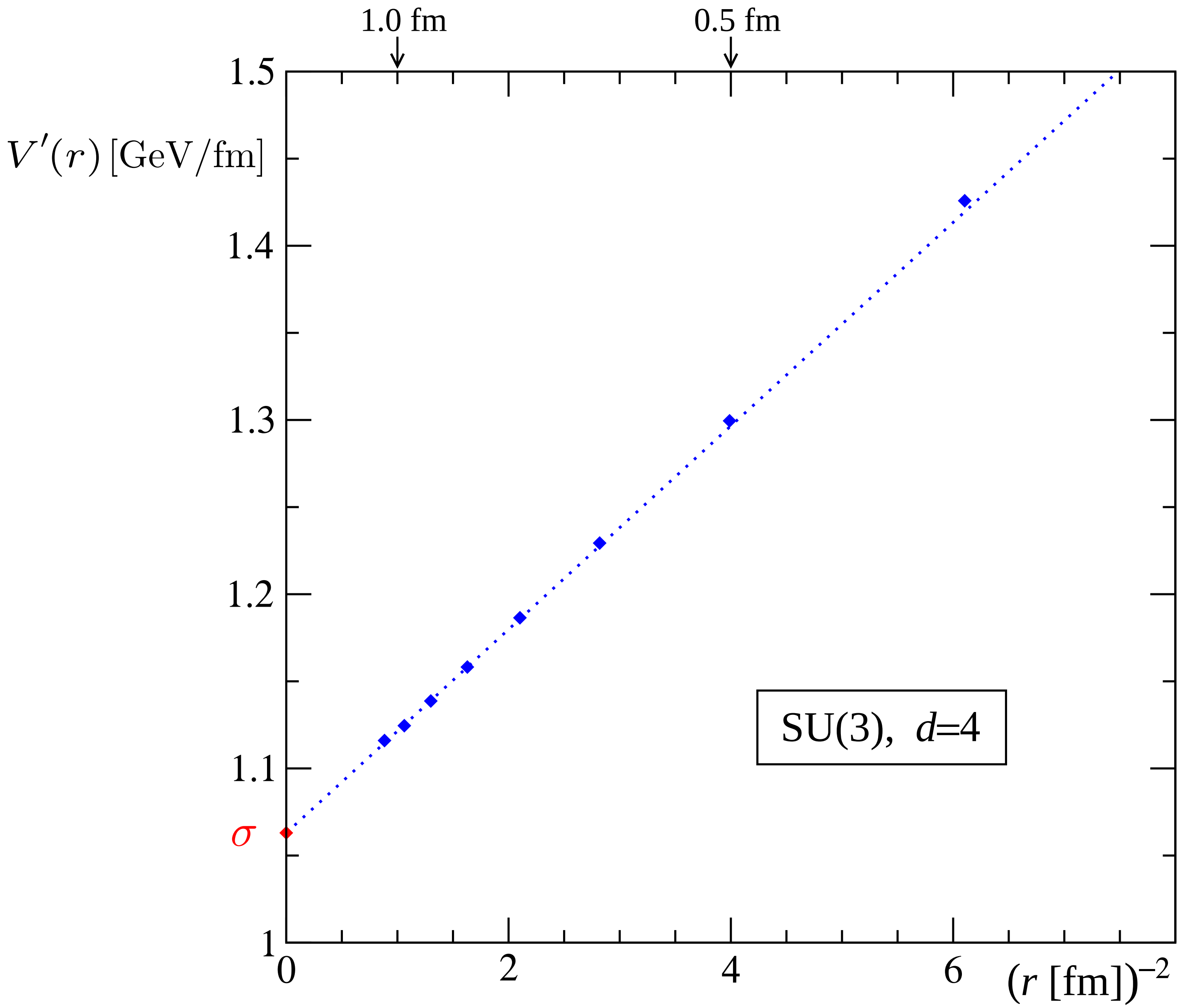
} First average $U^* \otimes U$ here for fixed b.c. and then take product

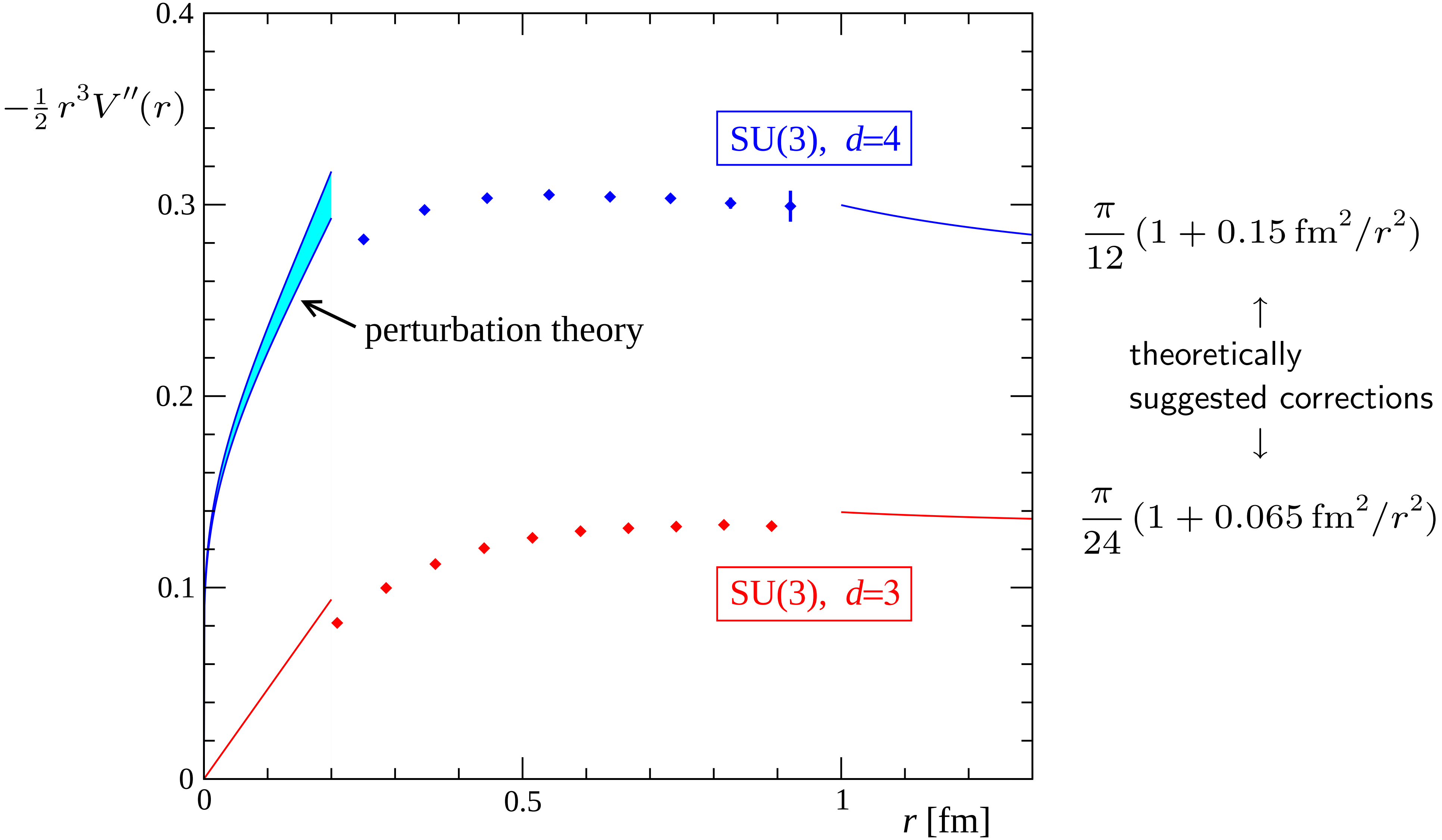
$$\langle P(r)^* P(0) \rangle = \langle \mathrm{tr}\{ [U^* \otimes U] \, [U^* \otimes U] \ldots [U^* \otimes U] \} \rangle$$

$$\uparrow$$

$$\sim \mathrm{e}^{-2\sigma r a}$$

$\Rightarrow$ exponential reduction of the statistical errors! M.L. & Weisz '01


1.0 fm
0.5 fm
1.5
$V'(r)$ [GeV/fm]
1.4
1.3
1.2
1.1
σ
1
0
2
4
6
$(r\ [\text{fm}])^{-2}$
SU(3), d=4



M.L. & P. Weisz, JHEP 07 (2002) 049 [hep-lat/0207003]

Excited states

Energy spectrum

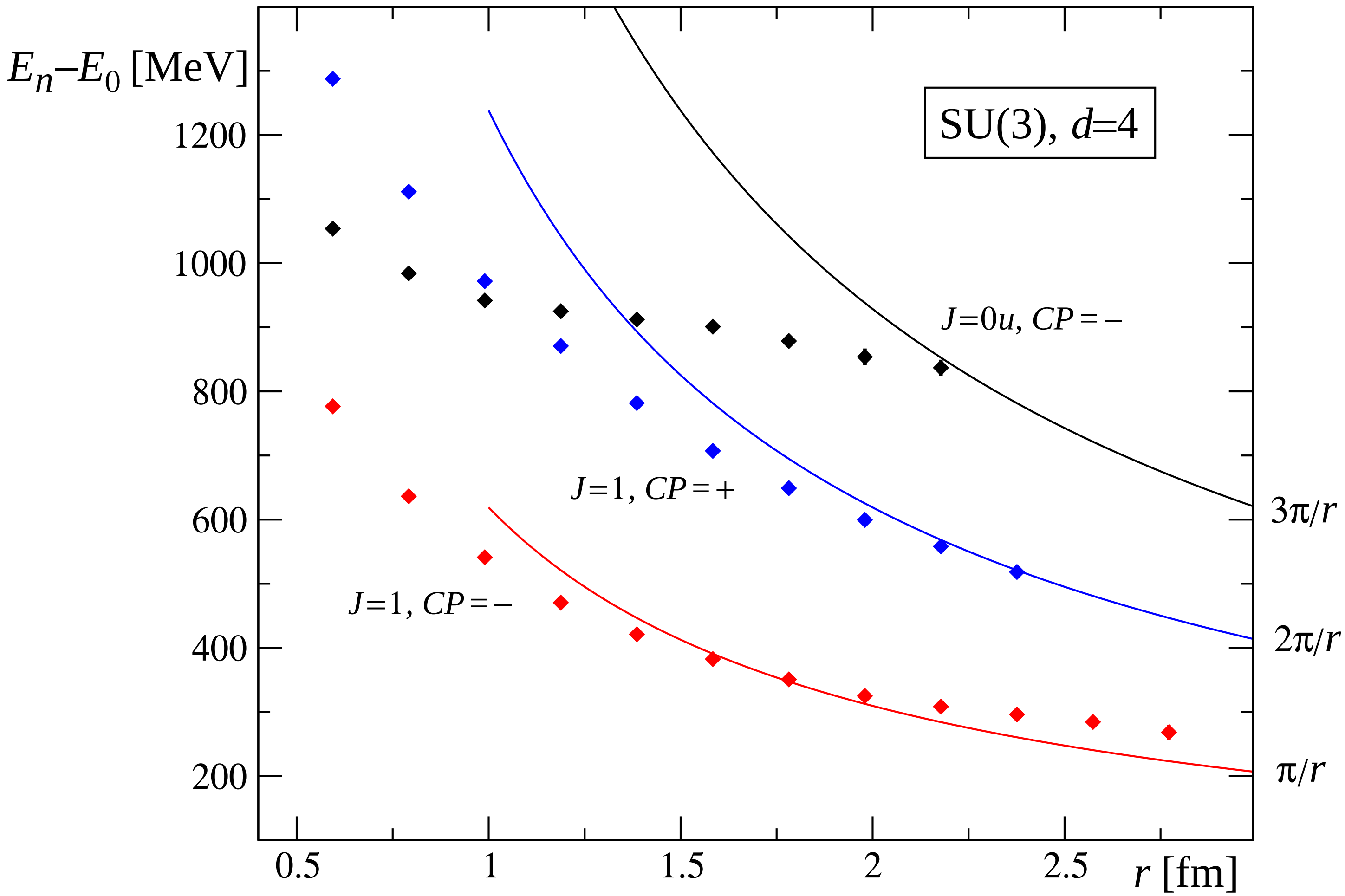
$$E_n = E_0 + \frac{n\pi}{r} + \mathrm{O}(r^{-2})$$

- Classified by axial symmetry
- Accidental degeneracies to this order

Spectrum is difficult to calculate in LGT

$\Rightarrow$ use variational techniques, anisotropic lattices, . . .

So far the multilevel algorithm has here only been used in preliminary studies



Juge, Kuti & Morningstar '02

String self-interaction effects

M.L. & P. Weisz, JHEP 0407 (2004) 014

The possible string interactions of dimension 4 are

$$S_2 = \int \mathrm{d}^2 z \left\{ \tfrac{1}{4} c_2 \left(\partial_a h \partial_a h\right) \left(\partial_b h \partial_b h\right) + \tfrac{1}{4} c_3 \left(\partial_a h \partial_b h\right) \left(\partial_a h \partial_b h\right) \right\}$$

$\Rightarrow$ energy spectrum (in $d = 4$)

$$E_0 = \sigma r + \mu - \frac{\pi}{12 r} \left\{ 1 + b/r + (b/r)^2 \right\} + \frac{\pi^2}{576 r^3} \left(2 c_2 + 3 c_3\right) + \mathrm{O}(r^{-4})$$

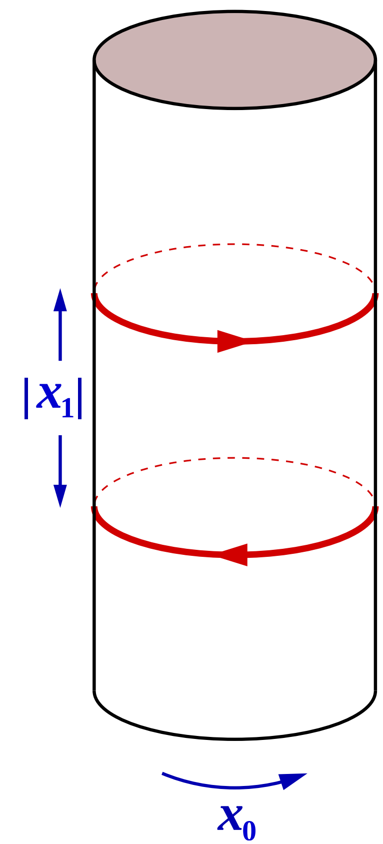
$$E_1 = E_0 + \frac{\pi}{r} \left\{ 1 + b/r + (b/r)^2 \right\} + \frac{\pi^2}{24 r^3} \left(34 c_2 + 27 c_3\right) + \mathrm{O}(r^{-4})$$

...

Open–closed string duality

The Polyakov loop correlation function satisfies

$$\int \mathrm{d}x_\perp \, \langle P(x)^* P(0) \rangle = \sum_{n \geq 0} |v_n|^2 \mathrm{e}^{-\widetilde{E}_n |x_1|}$$



Should also be so in the effective theory

$$\Rightarrow \quad b = 0, \qquad (d-2)c_2 + c_3 = \frac{d-4}{2\sigma}$$

Only one parameter is left **!**

Actually none in $d = 3$ **!**

How does one show this?

1. Radon transform

The mapping

$$f(r) \to \hat{f}(|x_1|) = \int \mathrm{d}x_\perp f(r)$$

is invertible. In $d = 4$, for example, $f(r) = (2\pi r)^{-1} \hat{f}\,'(r)$

⇒ open–closed string duality implies

$$\langle P(x)^* P(0) \rangle = \sum_{n=0}^{\infty} |v_n|^2 \, 2r \left(\frac{\tilde{E}_n}{2\pi r} \right)^{\frac{1}{2}(d-1)} K_{\frac{1}{2}(d-3)}(\tilde{E}_n r)$$

2. Free-string case

Using the *modular invariance* of the η-function

$$\langle P(x)^* P(0)\rangle = \mathrm{e}^{-\mu T}\left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)}\sum_{n=0}^{\infty} w_n \mathrm{e}^{-\tilde{E}_n^0 r}$$

$$\tilde{E}_n^0 \equiv \sigma T + \frac{4\pi}{T}\left\{-\tfrac{1}{24}(d-2)+n\right\}$$ (closed-string energies)

In any dimension this matches the leading terms in

$$\ldots = \sum_{n=0}^{\infty} |v_n|^2 \left(\frac{\tilde{E}_n}{2\pi r}\right)^{\frac{1}{2}(d-2)} \mathrm{e}^{-\tilde{E}_n r}\left\{1+\frac{(d-2)(d-4)}{8\tilde{E}_n r}+\cdots\right\}$$

3. Next-to-leading order

To first order in the boundary interaction term S_1

$$\langle P(x)^* P(0)\rangle =$$

$$\mathrm{e}^{-\mu T}\left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)}\sum_{n=0}^{\infty} w_n \left\{1 + b(\tilde{E}_n^0 - \sigma T) + \frac{b}{2r}(d-2)\right\}\mathrm{e}^{-\tilde{E}_n^0 r}$$

Does not match the expansion in Bessel functions unless $b = 0$

⋆ At second order the duality can be worked out similarly
$\Rightarrow$ relation between c_2 and c_3

⋆ There are string partition functions that fulfil the requirement of duality exactly **!**

Energy spectrum to second order

In $d = 4$ dimensions

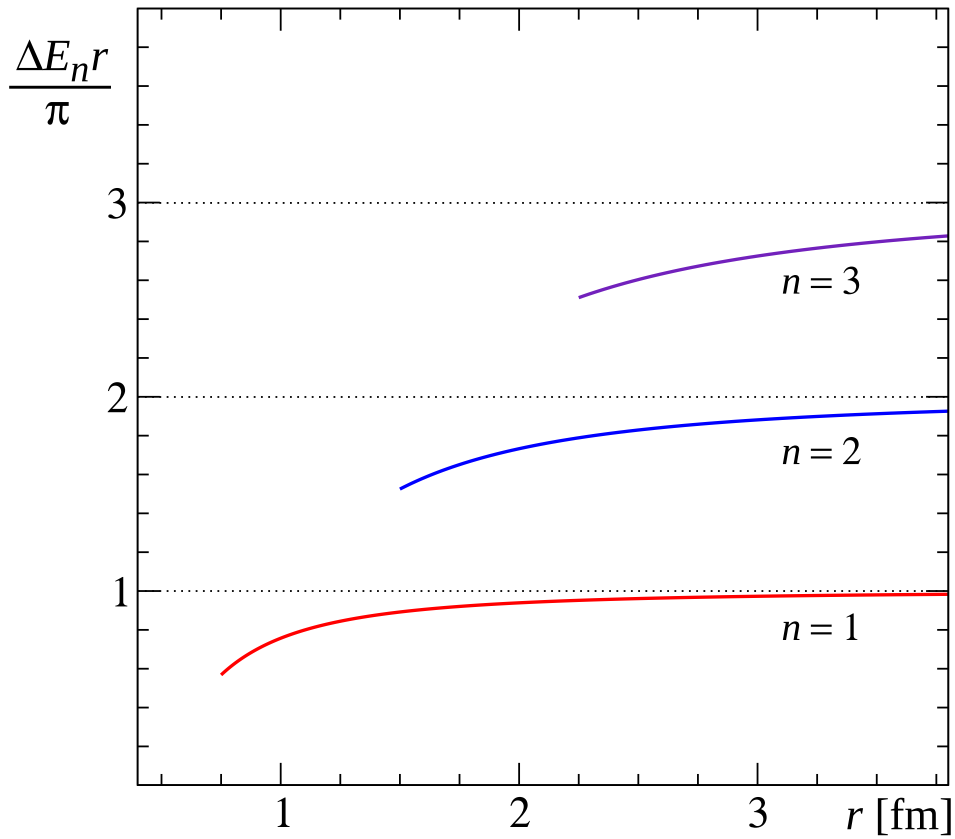
$$E_n = \sigma r + \mu + \frac{\pi}{r}\left(n - \tfrac{1}{12}\right) - c_2 \frac{\pi^2}{r^3}\left(n - \tfrac{1}{12}\right)^2 + \mathrm{O}(r^{-4})$$

- Degeneracies w_n are as in the free-string theory
- In $d = 3$ a similar formula holds with $c_2 = 1/2\sigma$ imposed by duality
- $c_2 = 1/2\sigma$ in the *classical* Nambu–Goto theory
- Higher-order terms get small only at $r \propto n$

Illustration

$$c_2 = \frac{1}{2\sigma} \simeq 0.093\,\mathrm{fm}^2$$

$$\Delta E_n \equiv E_n - E_0 = \frac{n\pi}{r} + \ldots$$



Conclusions

Effective string theory confirmed

- Central charge $= d - 2$
  $\Rightarrow$ excludes additional fermionic string modes
- String behaviour in $V(r)$ sets in at about $0.5\,\mathrm{fm}$

Excited states

- High-precision lattice simulations are now required
- Open–closed string duality $\leftrightarrow$ reparametrization invariance?

Fundamental "duality" QCD $\leftrightarrow$ string theory?

Polyakov '81ff, Maldacena '98, Witten '98, . . .