# From instantons to the Yang-Mills gradient flow

## History & resolution of a conceptual puzzle in non-perturbative QCD

*Martin Lüscher, CERN Physics Department*

**IFT Xmas Workshop, Universidad Autónoma de Madrid, 9.-11. December 2015**

## Introduction

*Discovery of instanton solutions in YM gauge theories* Belavin et al. '75

$\Rightarrow$ gauge fields carrying a non-zero value of the topological charge

$$Q = \int \mathrm{d}^4x \, q(x), \qquad q(x) = \frac{1}{32\pi^2} F^a_{\mu\nu}(x) {}^*F^a_{\mu\nu}(x),$$

may have important non-perturbative effects

*Axial anomaly* Adler '69; Bell & Jackiw '69

In QCD with $N_\mathrm{f}$ massless quarks, the axial anomaly

$$\partial_\mu j^5_\mu(x) = 2N_\mathrm{f} q(x), \qquad j^5_\mu = \text{flavour-singlet axial current},$$

provides a link to the physics of quarks

## *Witten-Veneziano formula*

Witten '79; Veneziano '79

In the large-$N_c$ limit

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2}\,\chi_t|_{N_f=0} + \text{subleading}$$

$$\chi_t = \int_x \langle q(x)q(0)\rangle = \frac{\langle Q^2\rangle}{V} \qquad \text{("topological susceptibility")}$$

⇒ Topological fluctuations give $\eta'$-meson a mass $\propto \Lambda_{\overline{\rm MS}}$

⇒ Resolves so-called $\mathrm{U}(1)_\mathrm{A}$ problem if $\chi_t|_{N_f=0} \simeq (190\,\mathrm{MeV})^4$

⇒ Would like to compute $\chi_t$ in lattice QCD

However, the 2-point function

$$\langle q(x)q(0)\rangle \underset{x\to 0}{\sim} (x^2)^{-4}$$

has a non-integrable short-distance singularity

$\Rightarrow$ its Fourier transform

$$\langle q(x)q(0)\rangle \underset{\mathrm{F.T.}}{=} c_0\Lambda^4 + c_1\Lambda^2 p^2 + c_2(p^2)^2 + (p^2)^3 \int_0^\infty \mathrm{d}s\, \frac{\rho(s)}{s^3(s+p^2)}$$

is determined only up to contact terms

$\Rightarrow$ $\chi_t = c_0\Lambda^4$ **is not unambiguously defined!**

The fields in 4d (Euclidean) QFT are typically nowhere continuous!

Colella & Lanford 73

And without continuity there is no topology . . .

- ⋆ *Do the topological sectors in field space exist beyond the semi-classical approximation?*
- ⋆ *Is there a sensible "universal" definition of* $\chi_t$*?*
- ⋆ *Can the WV formula be given an unambiguous meaning?*

## Milestones

**1981:** Smoothness & topology in lattice gauge theory.

**1992-98:** Lattice QCD with exact chiral symmetry.

**2002-04:** Universal "density-chain" formula for $\chi_t$. Rederivation of the WV formula.

**2010:** Finiteness of the Yang-Mills gradient flow. Emergence of the topological sectors in the continuum limit.

**2015:** Equivalence of the gradient-flow and density-chain definitions of $\chi_t$.

## Smoothness & topology in lattice gauge theory

ML '81

*Any lattice gauge field can be smoothly deformed to the trivial field*

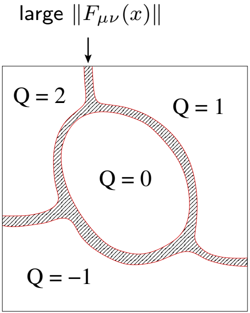
But the subspace of fields satisfying

$$\|F_{\mu\nu}(x)\| < \frac{1}{2a^2} \quad \text{for all } x,\mu,\nu$$

$\uparrow$

defined through plaquette loop

divides into disconnected charge sectors!




field space

**Rigorously proved through the construction of a local topological charge density**

- *The topological charge sectors are included in the lattice theory*

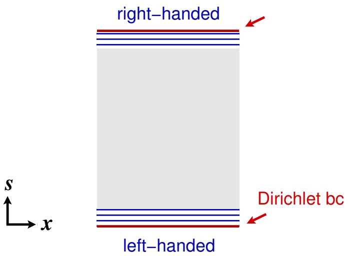
- *In the functional integral, the bound* $\|F_{\mu\nu}\| < 1/2a^2$ *however tends to be violated with high probability*

- *And there is no algebraic link to the lattice quark fields and the axial anomaly*

## Lattice QCD with exact chiral symmetry

Kaplan '92
Hasenfratz et al. '98
Neuberger '98
ML '98

4d Weyl from 5d massive Dirac fermions




Rubakov & Shaposhnikov '83
Callan & Harvey '85

- ⋆ Effect persists on a 5d lattice with Wilson-Dirac fermions and mass $\sim 1/a$
- ⋆ Even if coupled to an $s$-independent SU(3) gauge field
  $\Rightarrow$ 4d lattice QCD with exactly massless quarks!

Effective 4d theory

$$S = S_{\mathrm{G}} + a^4 \sum_x \overline{\psi}(x) D \psi(x)$$

$D$ : "Neuberger-Dirac operator"

$D$ satisfies the "Ginsparg-Wilson relation" Ginsparg & Wilson '82

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

$\Rightarrow$ the infinitesimal chiral transformations

$$\delta\psi = \lambda^a \gamma_5 (1 - aD)\psi, \qquad \delta\overline{\psi} = \overline{\psi}\gamma_5\lambda^a$$

leave the action invariant!

The flavour-singlet transformation is anomalous since $\mathrm{Tr}\{\gamma_5(1 - aD)\} \neq 0$

Moreover, there exists a local gauge-invariant axial current such that

$$\partial_\mu j^5_\mu(x) = 2N_\mathrm{f} q(x)$$

$$q(x) = -\frac{a}{2}\mathrm{tr}\{\gamma_5 D(x,x)\}$$

$$Q = a^4 \sum_x q(x) = \text{topological charge} = \text{index}(D)$$

holds exactly

- *Full set of exact* $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f})$ *chiral Ward identities*

- *As a function of the gauge field,* $D$ *is discontinous across the charge sector boundaries, but provably smooth if* $\|F_{\mu\nu}\| < c/a^2$

Hernández, Jansen & ML '98

## Density-chain formula for $\chi_t$

Giusti, Rossi & Testa '04
ML '04

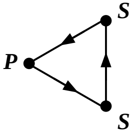
For any given gauge field

$$\mathrm{Tr}\{\gamma_5 f(D)\} = f(0) \times \text{index}(D)$$

since $(\psi, \gamma_5\psi) = 0$ if $D\psi = \lambda\psi$ and $\lambda \neq 0$

$\Rightarrow$ density-chain representations of $Q$

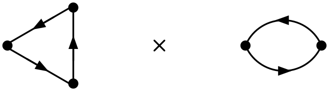
$$Q = \text{index}(D) = m^3\,\mathrm{Tr}\{\gamma_5 (D+m)^{-3}\}$$

$= m^3 \times$ 


$$= \text{Wick contraction of} \left\{ m^3 \int_{x,y,z} P_{12}(x) S_{23}(y) S_{31}(z) \right\}$$

$\Rightarrow$ If there are $N_{\mathrm{f}} \geq 5$ quark flavours with masses $m_1, \ldots, m_5$

$$\chi_t = m_1 \ldots m_5 \int_{x_1,\ldots,x_4} \langle P_{12}(x_1) S_{23}(x_2) S_{31}(x_3) P_{45}(x_4) S_{54}(0) \rangle$$



- ⋆ Short-distance singularities are all integrable
- ⋆ Provides a <u>universal</u> definition of $\chi_t$ (with $mP, mS$ canonically normalized)
- ⋆ May use valence instead of sea quarks $\Rightarrow$ formula for any $N_{\mathrm{f}} \geq 0$

*The WV formula holds with the density-chain definition of $\chi_t$*

In lattice QCD with exact chiral symmetry

Giusti et al. '02
Seiler '02

- The density-chain formula for $\chi_t$ coincides with the naive expression

$$\chi_t = a^4 \sum_x \langle q(x) q(0) \rangle$$

- $\chi_t = 0$ in massless QCD

- $\langle 0|q(0)|\eta' \rangle = m_{\eta'}^2 F_\pi / \sqrt{2N_\mathrm{f}}$ to leading order of $1/N_\mathrm{c}$

Assuming $m_{\eta'}^2 \sim 1/N_\mathrm{c}$, $F_\pi^2 \sim N_\mathrm{c}$, these properties imply the WV formula

*The WV formula has acquired an unambiguous meaning in this way!*

*But does the field space divide into disconnected topological sectors?*

## Yang-Mills gradient flow

Consider the "flow of fields" $B_\mu(t,x)$, $t \geq 0$, defined by

$$B_\mu|_{t=0} = A_\mu$$

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Atiyah & Bott '82, . . . [Morse theory of field space]

- *Purely geometric equation*
- *Global existence is rigorously guaranteed on the lattice*
- *The solution is a gauge-covariant function(al) of* $A_\mu$

The evolution in the "flow time" $t$ has a smoothing effect

$$\underbrace{\partial_t B_\mu = \Delta B_\mu}_{\text{heat equation}} - \partial_\mu \partial_\nu B_\nu + \text{non-linear}$$

Smoothing range $= \sqrt{8t}$

*Correlation functions of gauge-invariant fields like* ML & Weisz '11

$$q(t,x) = \frac{1}{32\pi^2} \left( G^a_{\mu\nu} {}^*G^a_{\mu\nu} \right)(t,x)$$

*do not require renormalization and have no short-distance singularities!*

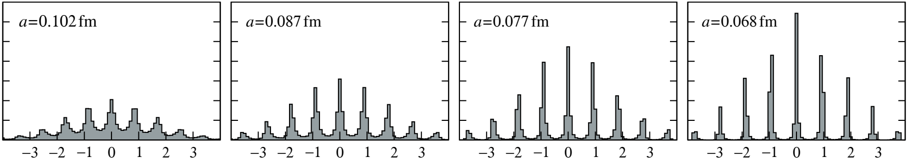
In particular, on the lattice ML '10

$\|G_{\mu\nu}(t,x)\| \ll 1/a^2$ as $a \to 0$ at fixed $t$

with probability $1 - \mathrm{O}(a^p)$

## Emergence of the topological sectors as $a \to 0$

Distribution of $Q(t)$ at fixed smoothing range $\sqrt{8t} = 0.34\,\mathrm{fm}$




Cè, Consonni, Engel & Giusti '15

### Interpretation

- ⋆ The field space "between the sectors" very rapidly gets suppressed as $a \to 0$
- ⋆ Theoretically expected in view of $\|G_{\mu\nu}(t,x)\| \ll 1/a^2$
- ⋆ The existence of the topological sectors is a dynamical property of QCD

## Gradient-flow formula for $\chi_t$

Does

$$\int_x \langle q(t,x)q(t,0)\rangle = \frac{\langle Q(t)^2\rangle}{V}$$

coincide with the density-chain $\chi_t$?

The expression is independent of $t$ since

$$\partial_t q = \frac{1}{8\pi^2}\partial_\mu\{\underbrace{\partial_t B_\nu^{a*} G_{\mu\nu}^a}_{\text{gauge invariant}}\}$$

$\Rightarrow$ can make contact with $\langle q(x)q(0)\rangle$ and the density chains by taking $t \to 0$

*The equality with the density-chain $\chi_t$ has been proved in the pure gauge theory*

*Probably true in QCD too* Cè, Consonni, Engel & Giusti '15

## Finally, a number!

Del Debbio et al. '05
ML & Palombi '10
Cichy et al. '15
Cè et al. '15

Using lattice gauge theory and the gradient flow

$$\chi_t|_{N_\mathrm{f}=0} = [190.6(1.6)\,\mathrm{MeV}]^4$$

Continuum limit taken, finite-volume effects controlled

Conversion to physical units using the Sommer radius $r_0 = 0.5\,\mathrm{fm}$

⇒ Almost too good — certainly large enough to resolve the $\mathrm{U}(1)_\mathrm{A}$ problem